Understanding and Addressing Algebra, Geometry, Rational Number, and Probability Misconceptions

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Friday, November 8, 2013

Session Objectives

Participants will:

- Learn about the underlying structure linking misconceptions across four content areas.
- Compare suggested pedagogical approaches with their current practice.
- Make connections between suggested pedagogical approaches for addressing misconceptions with CCSSM student learning objectives.

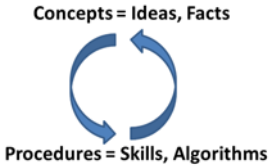
RESEARCH OVERVIEW

Differentiating the source of errors carries important consequences for intervention effectiveness

(Konold, 1988)

Learning Mathematics with Understanding

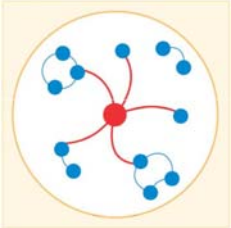
- Conceptual & Procedural Knowledge (Hiebert & Carpenter, 1992)



- Relational/Instrumental Understanding (Skemp, 1976/2006)
  - **Instrumental**: "Learning an increasing number of fixed plans, by which pupils can find their way from particular starting points (the data) to required finishing points (the answers to the questions)" (p. 14).
  - **Relational:** Building a conceptual structure ("schema") that can be used to produce an unlimited number of plans for getting from an starting point within the schema to any finishing point.

Internal and External Understanding

- **Internal Understanding**: "A mathematical idea or procedure or fact is understood if it is part of an internal network…if its mental representation is part of a network of representations" (Hiebert & Carpenter, 1992, p. 67).



- **External Understanding**: Connections between different representations of the same mathematical idea, fact, or procedure or the same representation of different ideas, facts, or procedures. (e.g., spoken language, written symbols, algebra blocks, graphs, tables)

What does understanding look like in the classroom?

Understanding in the Classroom

(Clement, 1982; Fisher, 1988; Resnick, 1983; Rosnick & Clement, 1979)

Observable Output to be Assessed	Underlying, Supporting Structure
Correct Solution	Correct Reasoning Applied Correctly to the Context
Correct Solution	Correct Reasoning Applied Incorrectly to the Context (or unrelated to context)
Correct Solution	Faulty Reasoning
Incorrect Solution	Correct Reasoning Applied Correctly to the Context (Careless Errors, Slips)
Incorrect Solution	Correct Reasoning Applied Incorrectly to the Context (i.e., Procedural Knowledge isolated from Conceptual Knowledge)
Incorrect Solution	Faulty Reasoning

An Example

(Lobato et al., 2010)

- *The clown walks 10 centimeters in 4 seconds. How far will the frog walk in 8 seconds if the frog travels at the same speed as the clown?*
- Students might be able to set up $\frac{10}{4} = \frac{x}{8}$ and even solve correctly to find that $x = 20$ **without having a clear understanding of ratio**.

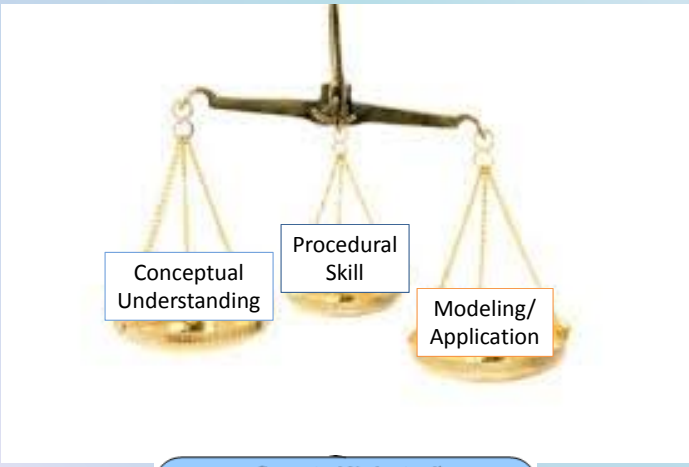
Deeper probing has found that some students who obtain a correct answer **believe that the frog is traveling faster than the clown because it is going further.** Such a misconception leaves students unprepared for problems stated in an unfamiliar format or more complex situations.

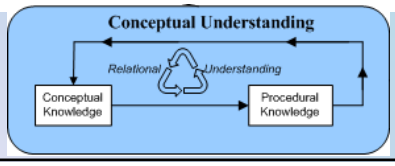
CCSSM Standards of Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning.

Page 6: http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

"Rigor" in the Common Core





Foundational Structures

- Images that support thinking and reasoning about quantitative relationships (Thompson, 1994).
- Critical for developing mental operations (i.e., thinking)
- Represent the highest level of Piaget's three types of images that people form as they develop understanding about a particular topic (Piaget & Inhelder, 1967).

Five Foundational Structures

(Rakes, 2010; Rakes & Ronau, under review)

- Rational Number Meaning (e.g., ratio, operator, quotient)
- Absolute/Relative Comparison (e.g., comparing ratios of quantities instead of quantities)
- Additive/Multiplicative Structures (e.g., recursive vs. functional relationships)
- Variable Meaning (e.g., beyond solving, what does the quantity represent?)
- Spatial Reasoning (e.g., van Hiele levels)

Foundational Structures in Learning Trajectories

(Lobato et al., 2010) (To view the Common Core Learning Trajectory Map, go to http://www.turnonccmath.net/

(1) Ratio as Composed Units

(2) Ratio as Multiplicative Comparisons

(3) Proportion as set of equivalent ratios (Absolute/Relative Comparisons)

(4) Students use iteration and partitioning to compose infinite sets of equivalent proportions

(5) Students use multiplication to abbreviate the process of iteration. 15 = 3 + 3+ 3 + 3 + 3 = 5 groups of 3

(6) Students recognize that a set of composed units are related through multiplication.

(7) Rate

(8) Students use Reasoning and sense making to develop a set of algorithms for solving proportion problems

(9) Students connect linearity to proportionality, that linear equations of the form $y = mx$ are statements of proportionality.

(10) Students recognize that $\frac{a}{b}$ is the same as a $\frac{1}{b}$ths.

Foundational Structures in Learning Trajectories

(Lobato et al., 2010)

(1) Ratio as Composed Units

Students recognize that every ratio is part of an infinite set of equivalent ratios, e.g., 1/3 = 2/6 = 3/9, etc.

(2) Ratio as Multiplicative Comparisons

Students understand that multiplication is more than repeated addition. 3 x 2 → adding 2 three times → 3 twos.

(3) Proportion as set of equivalent ratios (Absolute/Relative Comparisons)

Students understand which quantities are equal in a proportion and interpret ratios as fractions (2:3 is same as 2/3 of 1)

Example Pedagogical Strategies

- Targeting underlying reasoning
  - Questioning Strategies (Why? How Do You Know?)
  - Reflection (What? So What? Now What?)
  - Discussion Strategies (Think-Pair-Share)
- Assessment (Formative and Summative)
  - Tasks (all grade levels) target higher order thinking and high relevance/authenticity. (e.g., which is better rather than compute two values out of context)

Review of Pedagogical Strategies

Category	Description	Link
Assessment (Overall)	Formative and Summative Assessment to Drive Instruction	http://www.amle.org/BrowsebyTopic/Assessment/tabid/91/Default.aspx
Assessment Task Development	Fostering Higher Order Thinking in Authentic Contexts	http://www.leadered.com/rrr.html
Assessment Task Development	Task Rotation	http://www.ascd.org/publications/books/110129.aspx
Diversity/ Differentiation	Multiple Intelligences	http://howardgardner.com/multiple-intelligences/
Diversity/ Differentiation	Universal Design for Learning	• http://www.udlcenter.org/ • http://www.cast.org/udl/index.html
Instructional Approach/ Philosophy	Chunking	http://www.skillstoolbox.com/career-and-education-skills/learning-skills/effective-learning-strategies/chunking/
Instructional Approach/ Philosophy	Inquiry Based Learning	http://www.teachinquiry.com/index/Introduction.html
Instructional Approach/ Philosophy	Scaffolding	http://www.edutopia.org/blog/scaffolding-lessons-six-strategies-rebecca-alber
Instructional Strategy--Engaging Prior Learning	K-W-L	http://www.readwritethink.org/classroom-resources/printouts/chart-a-30226.html
Instructional Strategy--Interpersonal Learning	Cooperative Learning	http://www.cehd.umn.edu/research/highlights/coop-learning/
Instructional Strategy--Interpersonal Learning	Gallery Walks	http://serc.carleton.edu/introgeo/gallerywalk/index.html
Instructional Strategy--Intra- and Inter-personal Learning	Think-Pair-Share	http://serc.carleton.edu/introgeo/interactive/tpshare.html
Instructional Strategy—Key Words	Frayer Models	http://www.readingeducator.com/strategies/frayer.htm

Collaborative Groups

	Rational Number Meaning	Absolute/ Relative Comparisons	Additive/ Multiplicative Structure	Variable Meaning	Spatial Reasoning
Algebra	A	B	C	D	E
Geometry	F	G	H	I	J
Probability	K	L	M	N	O
Rational Number	P	Q	R	S	T

- Introduce Yourselves
- Read the Prompt
- Think about your responses individually to the following questions:
  1. What errors might students make on the problem? Can these errors be traced back to a misconception about the foundational structure? Why or why not?
  2. Is it possible for students to obtain a correct answer on the problem while having a misconception about the foundational structure? Why or why not?
  3. How might the teacher recognize and help students address the misconception(s)?

This Presentation will be available at
http://csrakes.yolasite.com/resources.php

THANK YOU!

What? So What? Now What?

(Silver, Strong, Perini, 2001, p. 156)

- Teacher provides and experience for students:
  - Students describe WHAT happened during the experience
  - Students ask themselves SO WHAT? and reflect on the learning from the experience
  - Students ask themselves NOW WHAT? and consider ways to extend the learning to other situations

Explaining Solutions on Word Problems

(Silver, Strong, Perini, 2001, p. 156)

1. Teacher selects word problem
2. Students read the problem carefully, underlining the question and the information needed to develop a solution.
3. Students solve the problem.
4. Students reflect on the process and list, in chronological order, the steps they followed.
5. With the teacher's help, students use transitional words (e.g., first, next, then, finally) to convert their steps into a paragraph. (Over time, students become independent.)
6. Students reread their explanations and ask:
    1. Are all the steps included?
    2. Are the steps in order?
    3. Do any terms need to be clarified?
    4. Are the transitional words well-chosen?
7. Students revise their explanations if necessary.

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Group A
Algebra and Rational Number Meaning

The temperature in degrees Celsius can be found by subtracting 32 from the temperature in degrees Fahrenheit and multiplying the result by $\frac{5}{9}$. If the temperature of a furnace is 393 degrees Fahrenheit, what is it in degrees Celsius, to the nearest degree?

A) 650 B) 1805 C) 40 D) 201 E) 72

Group B
Algebra and Absolute/Relative Comparisons

A flea, usually less than an eighth of an inch long, can long jump about thirteen inches and high jump about eight inches. For a human 5.5 feet tall, what is the minimum distance needed to jump proportionally at least as far as a flea?

Group C
Algebra and Additive/Multiplicative Structures

In the equation $y = 4x$, if the value of x is increased by 2, what is the effect on the value of y?

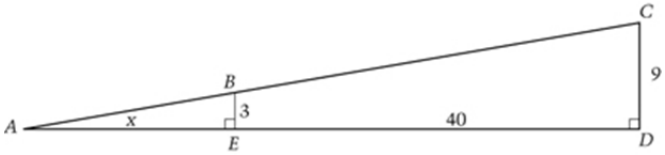
A) It is 8 more than the original amount.
B) It is 6 more than the original amount.
C) It is 2 more than the original amount.
D) It is 16 times the original amount.
E) It is 8 times the original amount.

Group D
Algebra and Variable Meaning

Cakes cost c dollars each and buns cost b dollars each. If I buy 4 cakes and 3 buns, what does $4c + 3b$ stand for?

Group E
Algebra and Spatial Reasoning

The figure below shows two right angles. The length of *AE* is *x* and the length of *DE* is 40. Show all of the steps that lead to finding the value of *x*. Your last step should give the value of *x*.



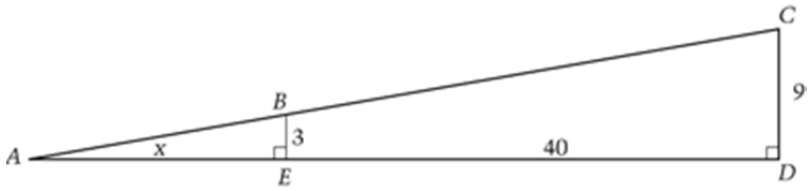
Group F
Geometry and Absolute/Relative Comparisons

The figure below shows a shaded portion of a rectangle and another shaded portion of a circle. How might these two shaded portions be considered equal?

Group G

Geometry and Absolute/Relative Comparison

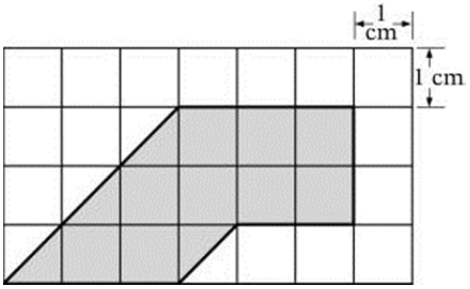
The figure below shows two right angles. The length of *AE* is *x* and the length of *DE* is 40. Show all of the steps that lead to finding the value of *x*. Your last step should give the value of *x*.



Group H

Geometry and Additive/Multiplicative Structure

What is the area of the shaded figure?



A) 9 square centimeters
B) 11 square centimeters
C) 13 square centimeters
D) 14 square centimeters

Group I

Geometry and Variable Meaning

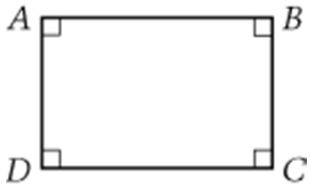
A scale drawing of a rectangular room is 5 inches by 3 inches. If 1 inch on this scale drawing represents 3 feet, what are the dimensions of the room?

A) 5 feet by 3 feet
B) 5 feet by 9 feet
C) 15 feet by 3 feet
D) 15 feet by 5 feet
E) 15 feet by 9 feet

Group J

Geometry and Spatial Reasoning

Sara was asked to draw a parallelogram. She drew the figure below.



Is Sara's figure a parallelogram? Why or why not?

Group K

Probability and Rational Number Meaning

The table below shows the gender and color of 7 puppies. If a puppy selected at random from the group is brown, what is the probability it is a male?

	Male	Female
Black	1	2
Brown	1	3

A) $\frac{1}{4}$ B) $\frac{2}{7}$ C) $\frac{1}{3}$ D) $\frac{1}{2}$ E) $\frac{2}{3}$

Group L
Probability and Absolute/Relative Comparison

A person is going to pick one marble without looking. For which dish is there the greatest probability of picking a black marble?

A B C D E

Group M
Probability and Additive/Multiplicative Structure

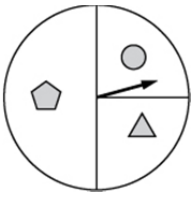
Twelve skiers compete in the final round of the Olympic freestyle skiing competition. How many different top three winners are possible? (Gold, Silver, Bronze).

Group N
Probability and Variable Meaning

Of the following two subGroups, which is larger? (a) Unmarried physicians, or (b) Unmarried physicians who like to travel abroad

Group O
Probability and Spatial Reasoning

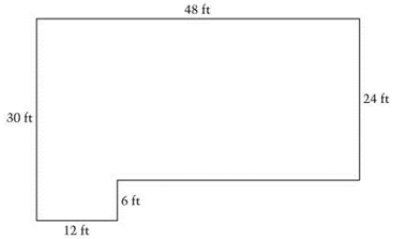
If Rose spins a spinner like the one below 300 times, about how many times should she expect it to land on the space with a circle?



A) 75 B) 90 C) 100 D) 120 E) 150

Group P
Rational Number Content and Rational Number Meaning

If you were to redraw the diagram using a scale of $\frac{3}{4}$ inch = 10 feet, what would be the length of the side that is 48 feet?



A) 3.0 in B) 3.6 in C) 5.6 in D) 7.5 in E) 12.0 in

Group Q

Rational Number and Absolute/Relative Comparison

In which of the following are the three fractions arranged from least to greatest?

A) $\frac{2}{7}, \frac{1}{2}, \frac{5}{9}$ B) $\frac{1}{2}, \frac{2}{7}, \frac{5}{9}$ C) $\frac{1}{2}, \frac{5}{9}, \frac{2}{7}$ D) $\frac{5}{9}, \frac{1}{2}, \frac{2}{7}$ E) $\frac{5}{9}, \frac{2}{7}, \frac{1}{2}$

Group R

Rational Number and Additive/Multiplicative Structures

The cost to mail a first-class letter is 33 cents for the first ounce. Each additional ounce costs 22 cents. (Fractions of an ounce are rounded up to the next whole ounce.)

How much would it cost to mail a letter that weighs 2.7 ounces?

A) 55 cents B) 66 cents C) 77 cents D) 88 cents E) 99 cents

Group S

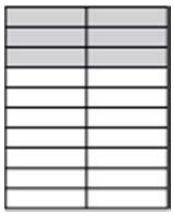
Rational Number and Variable Meaning

Jo has two snakes, String Bean and Slim. Right now, String Bean is 4.1 feet long and Slim is 5.4 feet long. Jo knows that two years from now, both snakes will be fully grown...At her full length, String Bean will be 7.3 feet long, while Slim's length when he is fully grown will be 8.6 feet. Over the next two years, will both snakes grow the same amount? Why or why not?

Group T

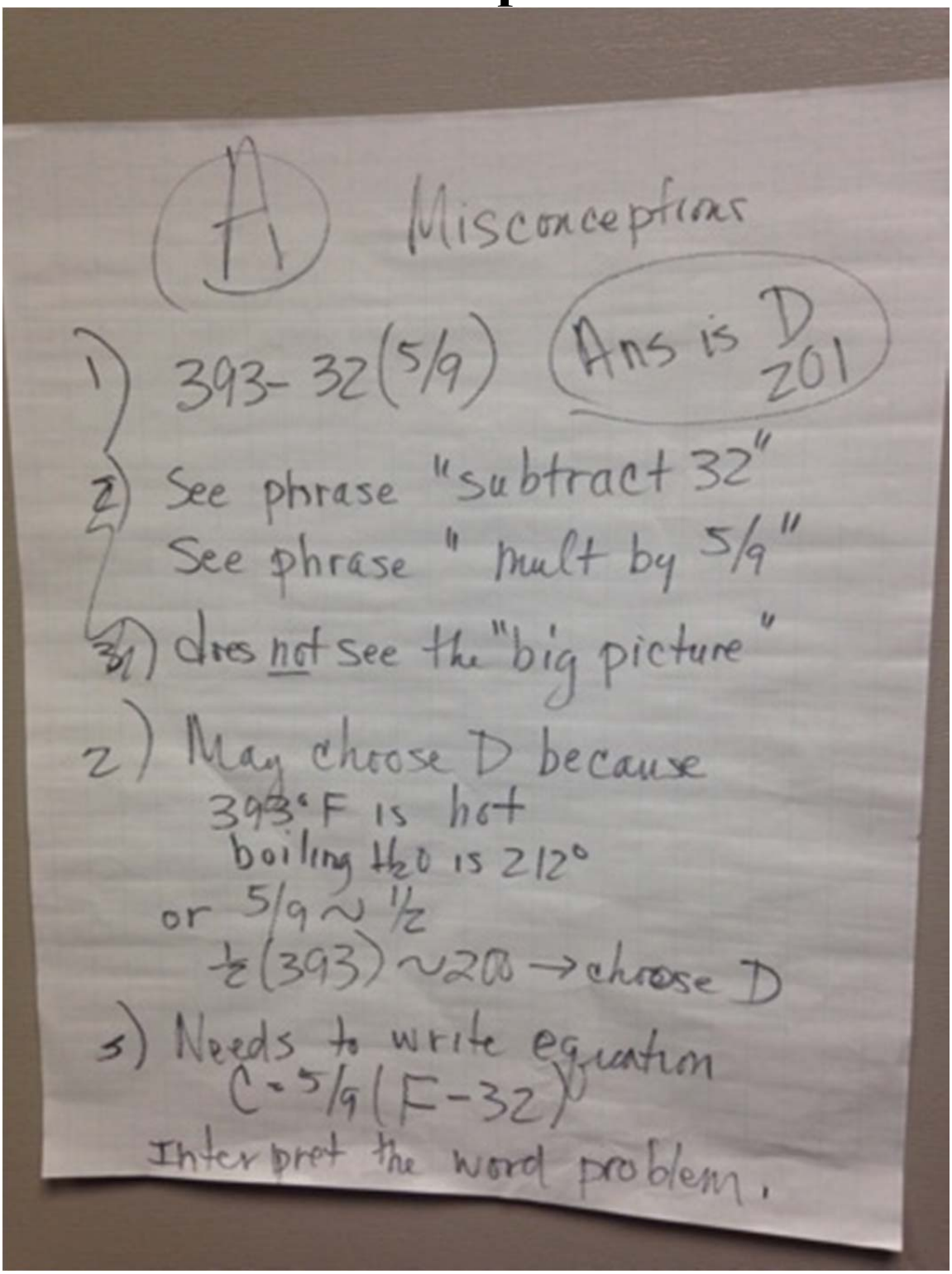
Rational Number and Spatial Reasoning

What fraction of the figure below is shaded?



A) $\frac{1}{4}$ B) $\frac{3}{10}$ C) $\frac{1}{3}$ D) $\frac{3}{7}$ E) $\frac{7}{10}$

Group A



A Misconceptions

1) 393 - 32(5/9) Ans is D 201

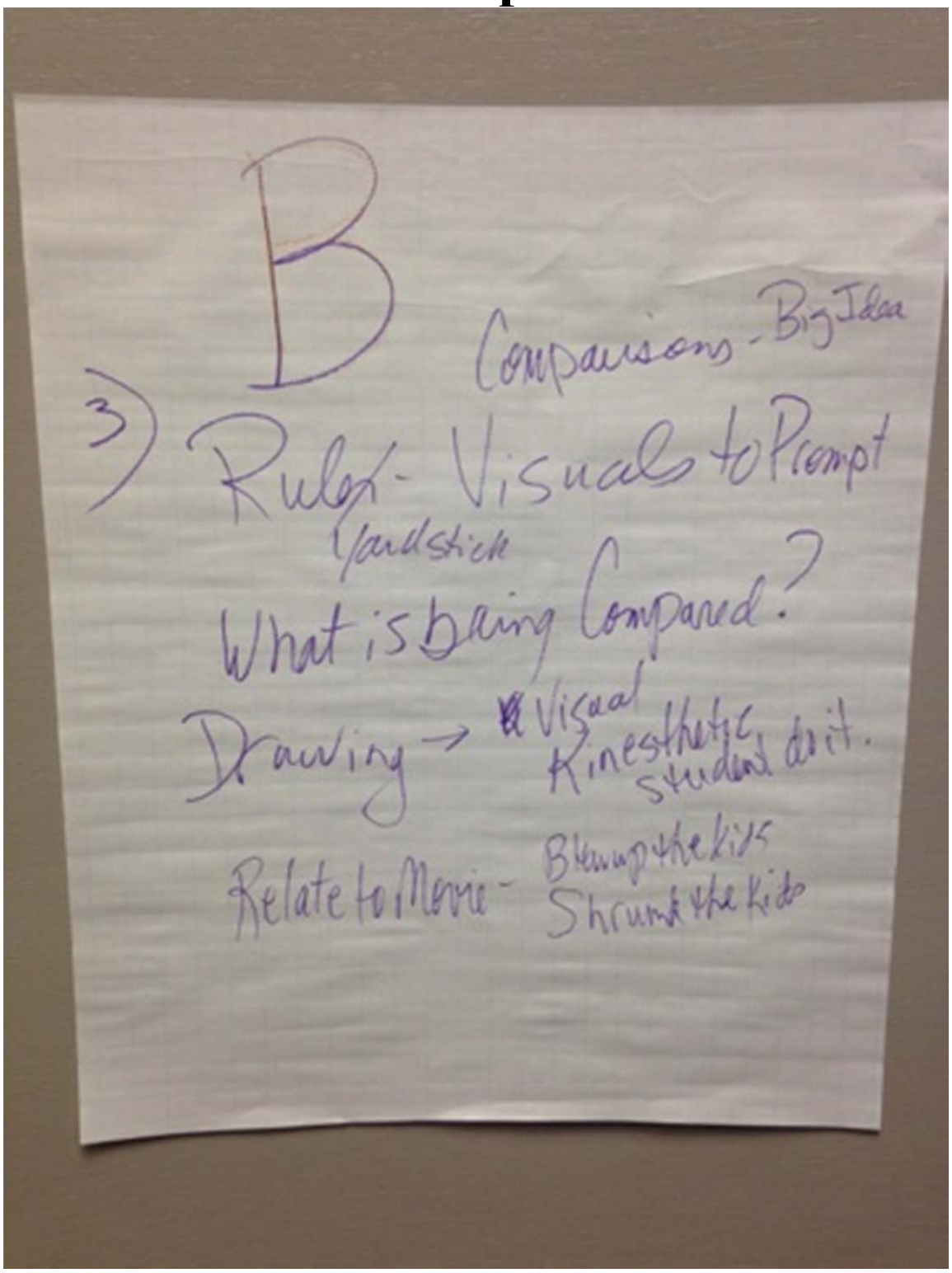
2) See phrase "subtract 32"
See phrase " mult by 5/9"

3) does not see the "big picture"

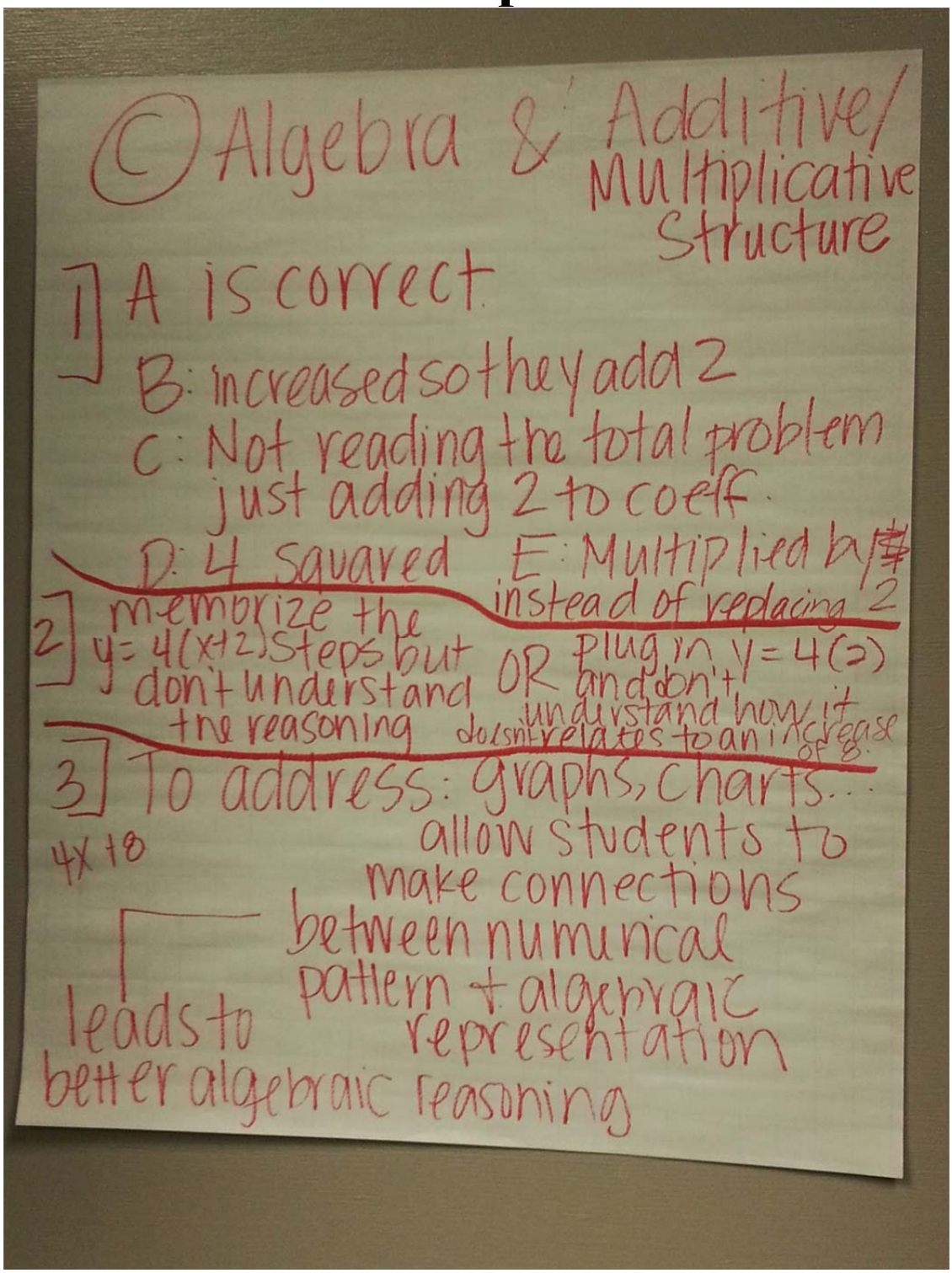
2) May choose D because
393°F is hot
boiling H_2O is 212°
or 5/9 ~ 1/2
1/2(393) ~ 200 → choose D

3) Needs to write equation
C = 5/9(F - 32)
Interpret the word problem.

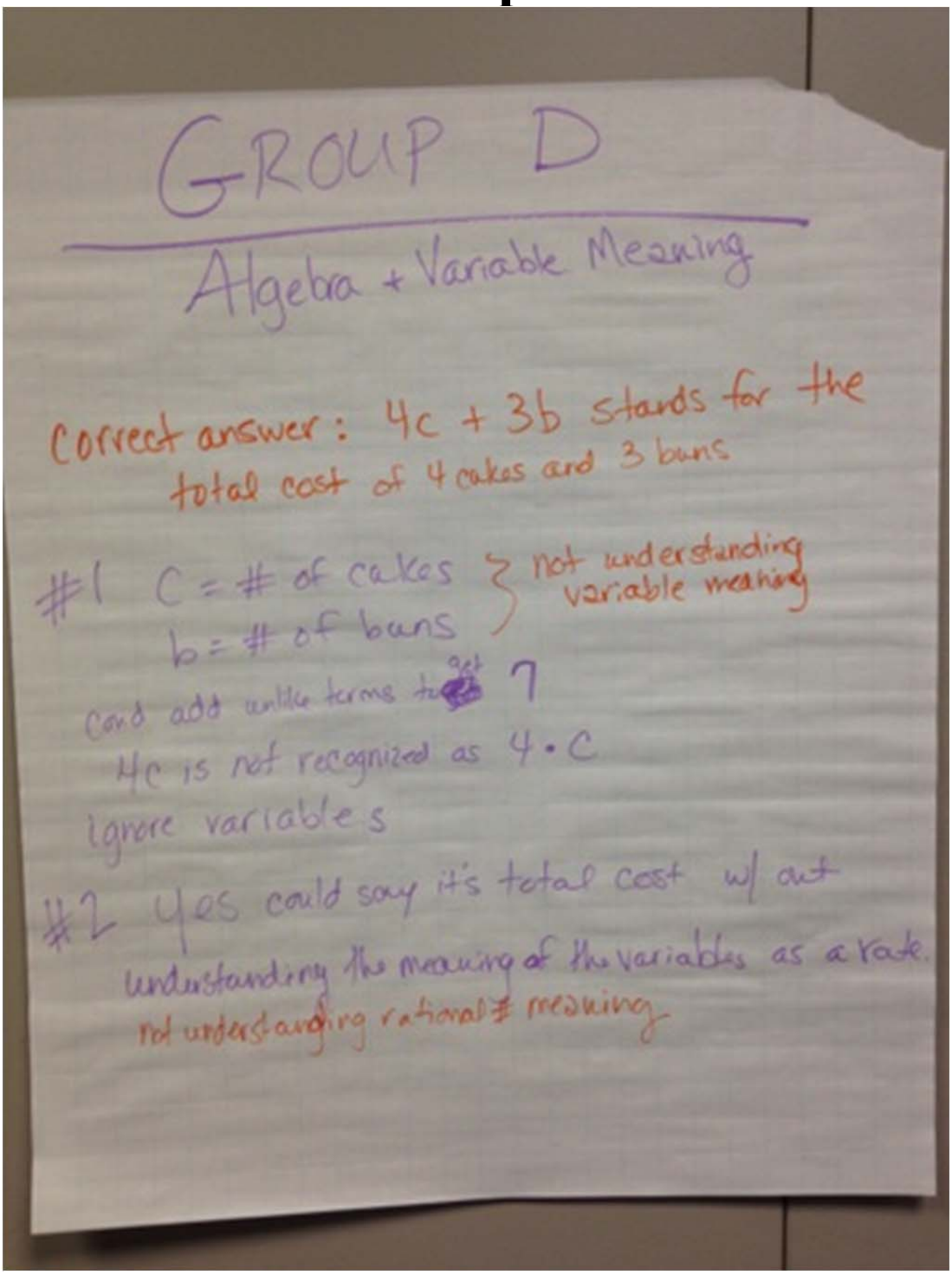
Group B


B
Comparisons - Big Idea
3) Rules - Visuals to Prompt
Yardstick
What is being Compared?
Drawing → Visual
Kinesthetic
student do it.
Relate to Movie - Blow up the kids
Shrunk the Kids

Group C


© Algebra & Additive/ Multiplicative Structure
1] A is correct.
B: increased so they add 2
C: Not reading the total problem just adding 2 to coeff.
D: 4 squared
E: Multiplied by # instead of replacing 2
2] memorize the y=4(x+2) steps but don't understand the reasoning
OR plug in y=4(2) and don't understand how it relates to an increase of 8.
3] To address: graphs, charts... allow students to make connections between numerical pattern + algebraic representation
4x+8
leads to better algebraic reasoning

Group D



GROUP D

Algebra + Variable Meaning

Correct answer: 4c + 3b stands for the total cost of 4 cakes and 3 buns

#1 c = # of cakes
b = # of buns
} not understanding variable meaning

Can't add unlike terms to get 7

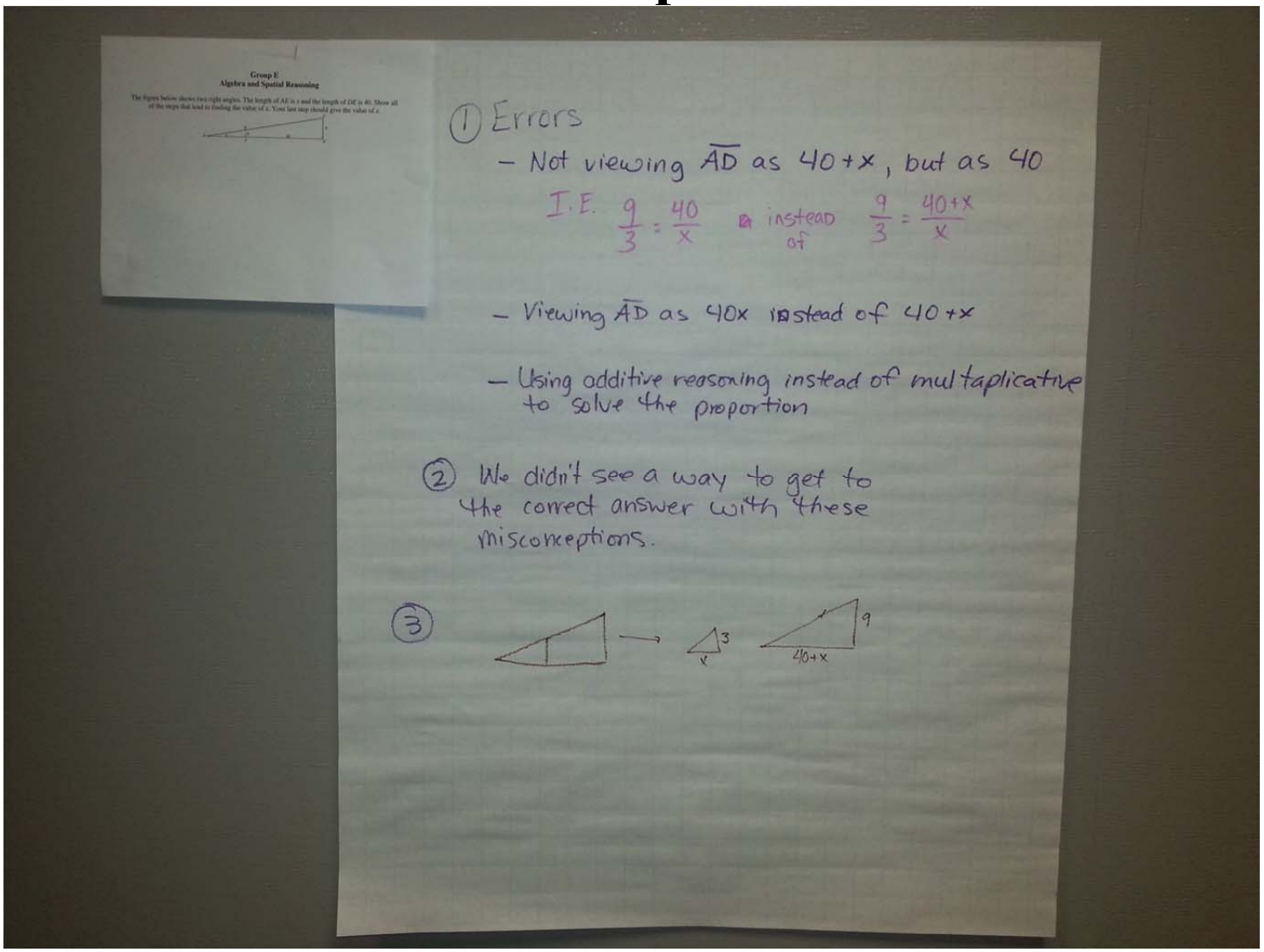
4c is not recognized as 4 • c

ignore variables

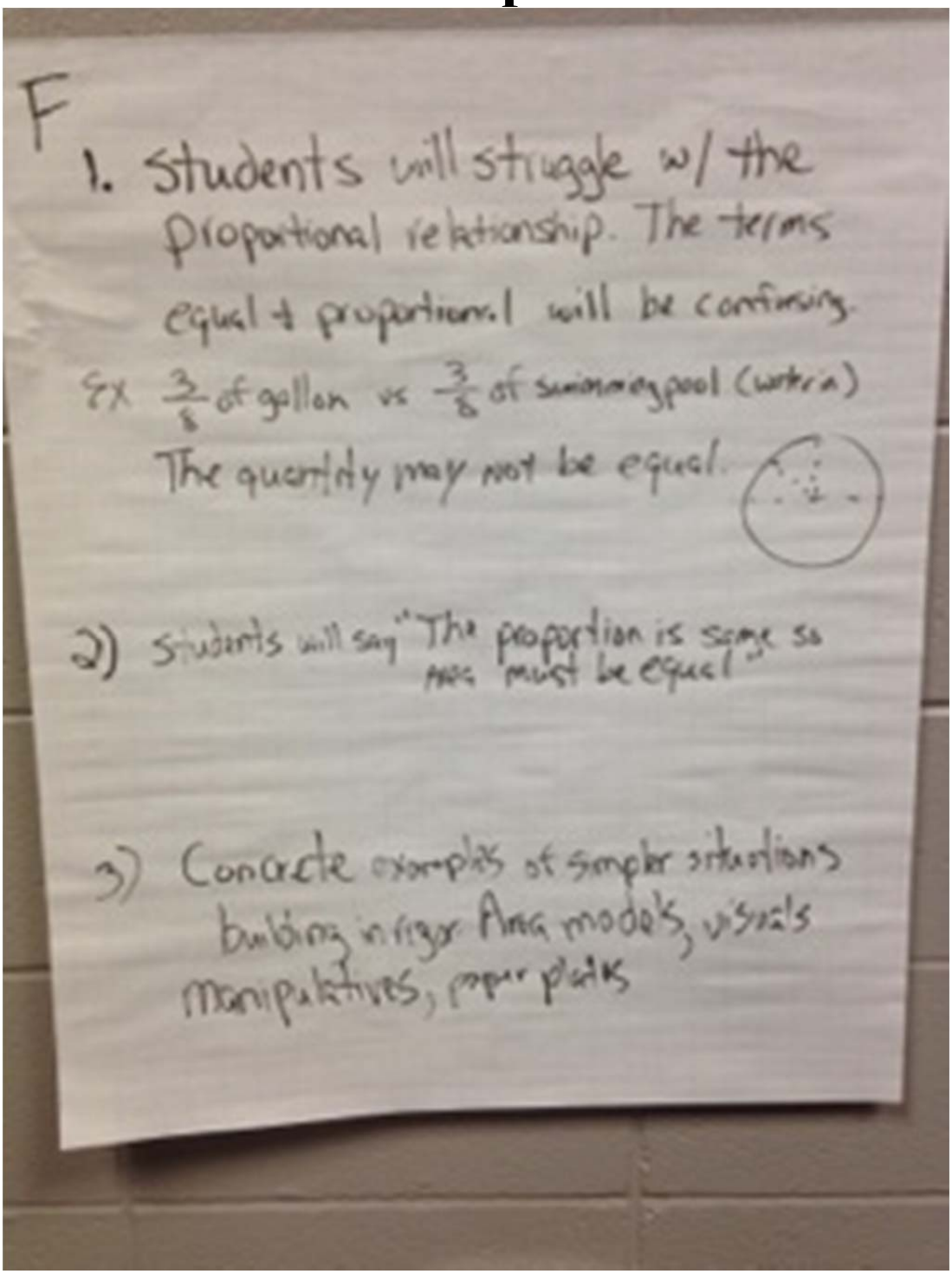
#2 yes could say it's total cost w/ out understanding the meaning of the variables as a rate.

not understanding rational # meaning

Group E


Group E
Algebra and Spatial Reasoning
The figure below shows two right angles. The length of AE is x and the length of DE is 40. Show all of the steps that lead to finding the value of x. Your last step should give the value of x.
① Errors
- Not viewing $\overline{AD}$ as $40+x$, but as 40
I.E. $\frac{9}{3} = \frac{40}{x}$ instead of $\frac{9}{3} = \frac{40+x}{x}$
- Viewing $\overline{AD}$ as $40x$ instead of $40+x$
- Using additive reasoning instead of multaplicative to solve the proportion
② We didn't see a way to get to the correct answer with these misconceptions.
③
3
x
9
40+x

Group F



F

1. Students will struggle w/ the proportional relationship. The terms equal + proportional will be confusing.

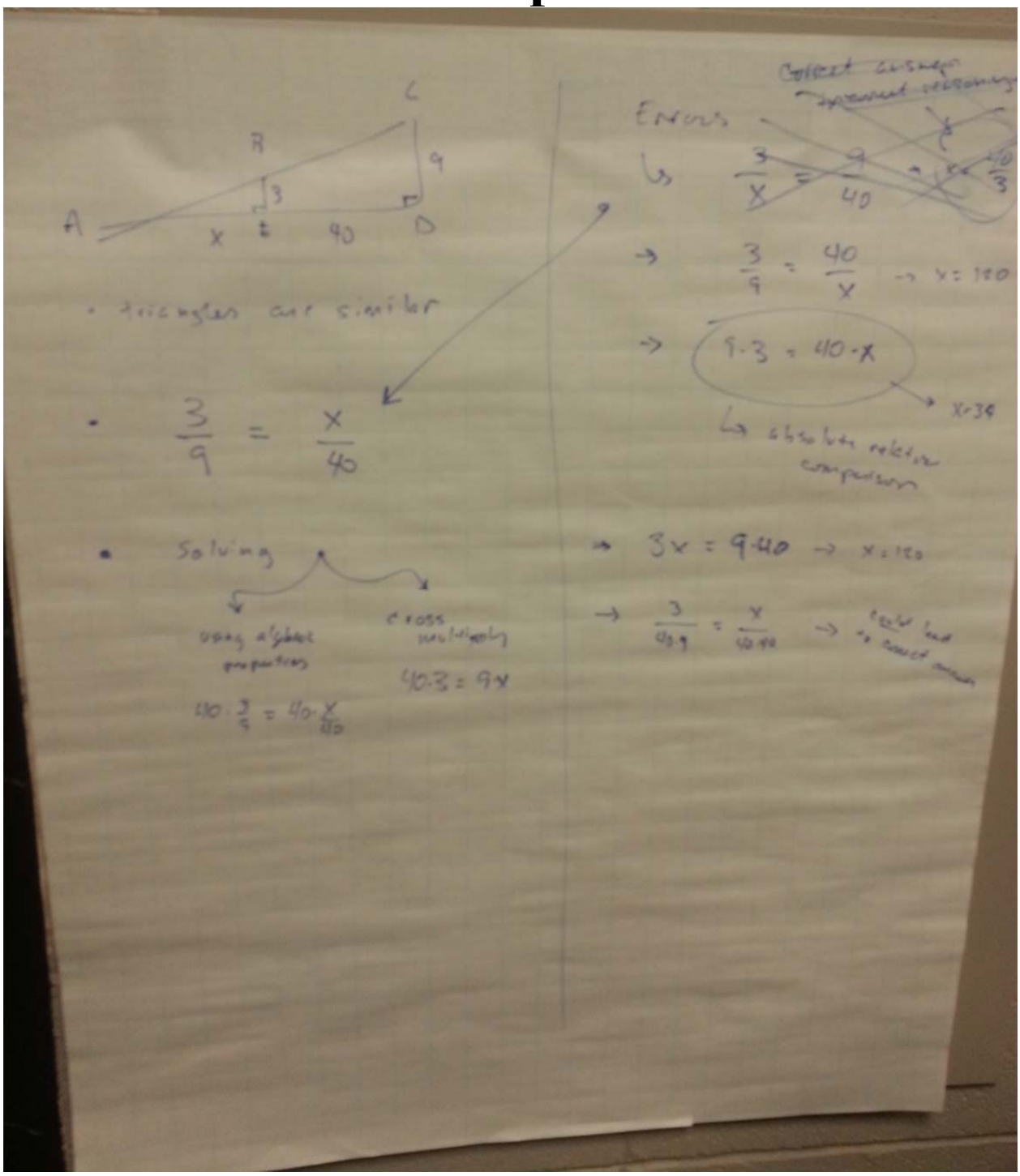
Ex $\frac{3}{8}$ of gallon vs $\frac{3}{8}$ of swimmingpool (water)

The quantity may not be equal.

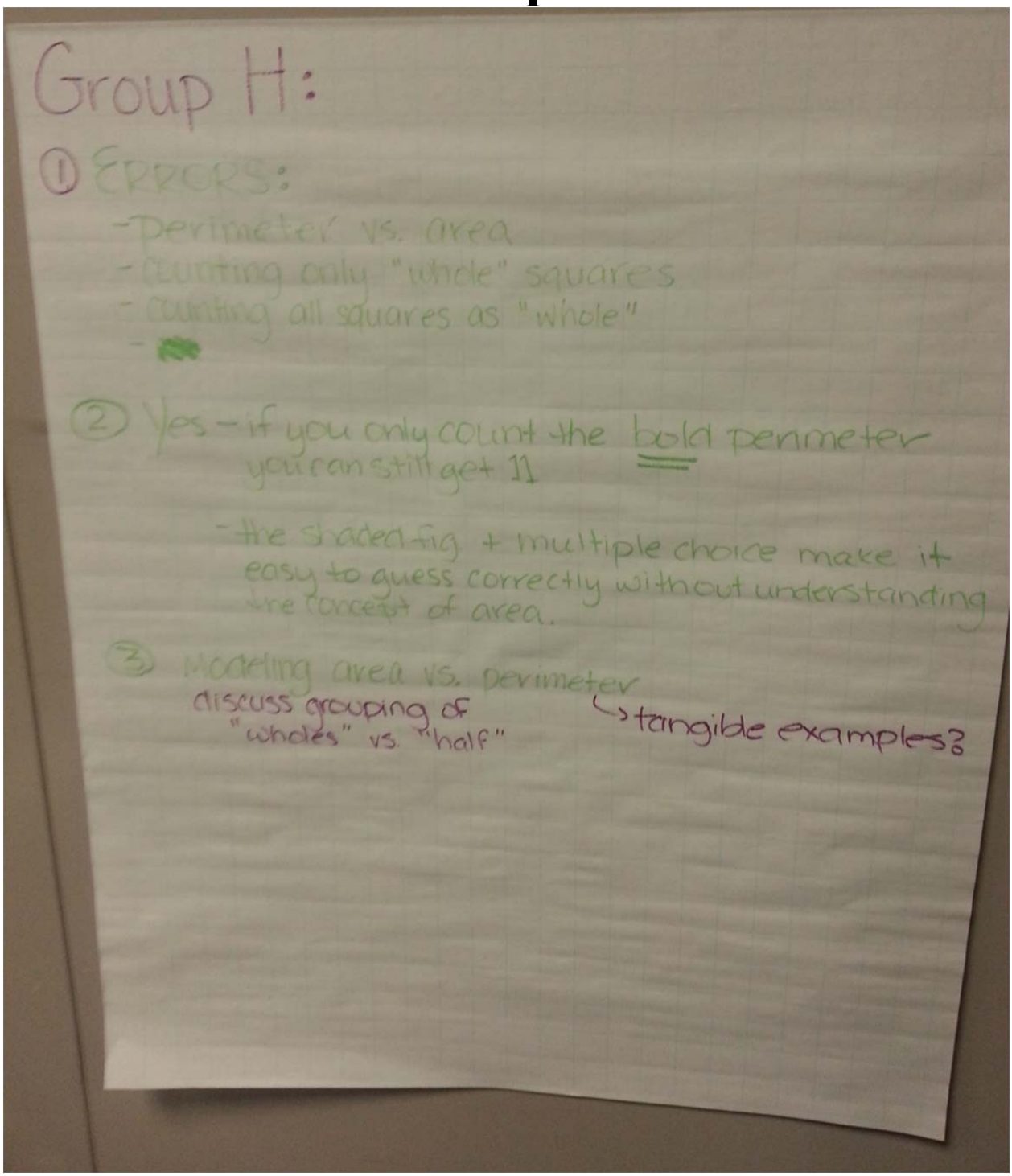
2) Students will say "The proportion is same so area must be equal"

3) Concrete examples of simpler situations building in rigor. Area models, visuals manipulatives, paper plates

Group G


C
B
9
3
A
x
40
D
• triangles are similar
• 3/9 = x/40
• Solving
using algebraic properties
cross multiplying
40·3 = 9·x
40·3/9 = 40·x/40
Errors
3/x = 9/40
3/9 = 40/x → x = 120
9·3 = 40·x
x=34
absolute relative comparison
3x = 9·40 → x = 120
3/(40·9) = x/(40·40)

Group H



Group H:

① ERRORS:

- perimeter vs. area
- counting only "whole" squares
- counting all squares as "whole"
- [illegible]

② Yes – if you only count the bold perimeter you can still get 11

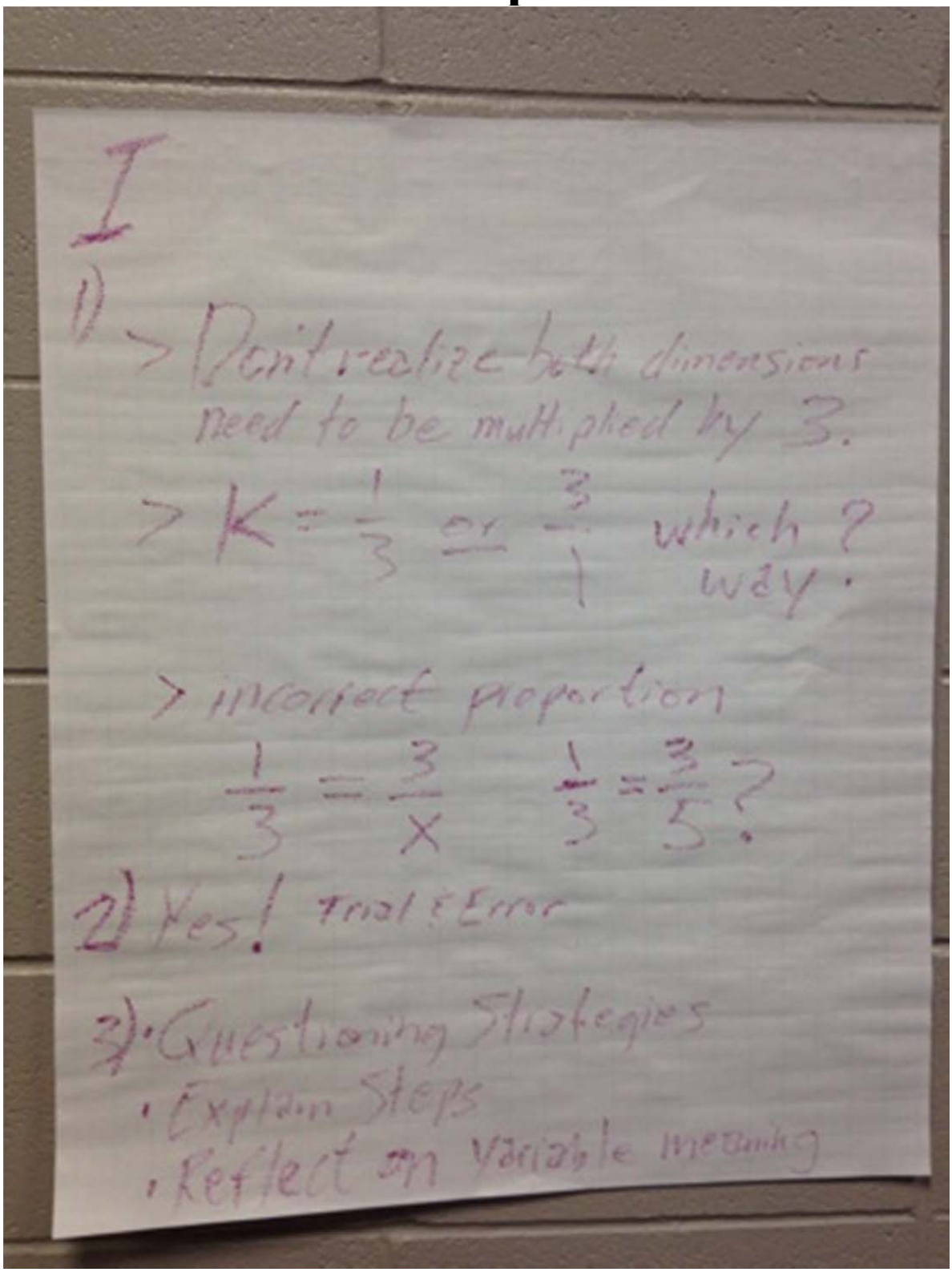
- the shaded fig + multiple choice make it easy to guess correctly without understanding the concept of area.

③ Modeling area vs. perimeter

discuss grouping of "wholes" vs. "half"

↳ tangible examples?

Group I



I

1) > Don't realize both dimensions need to be multiplied by 3.

> $K = \frac{1}{3}$ or $\frac{3}{1}$ which way?

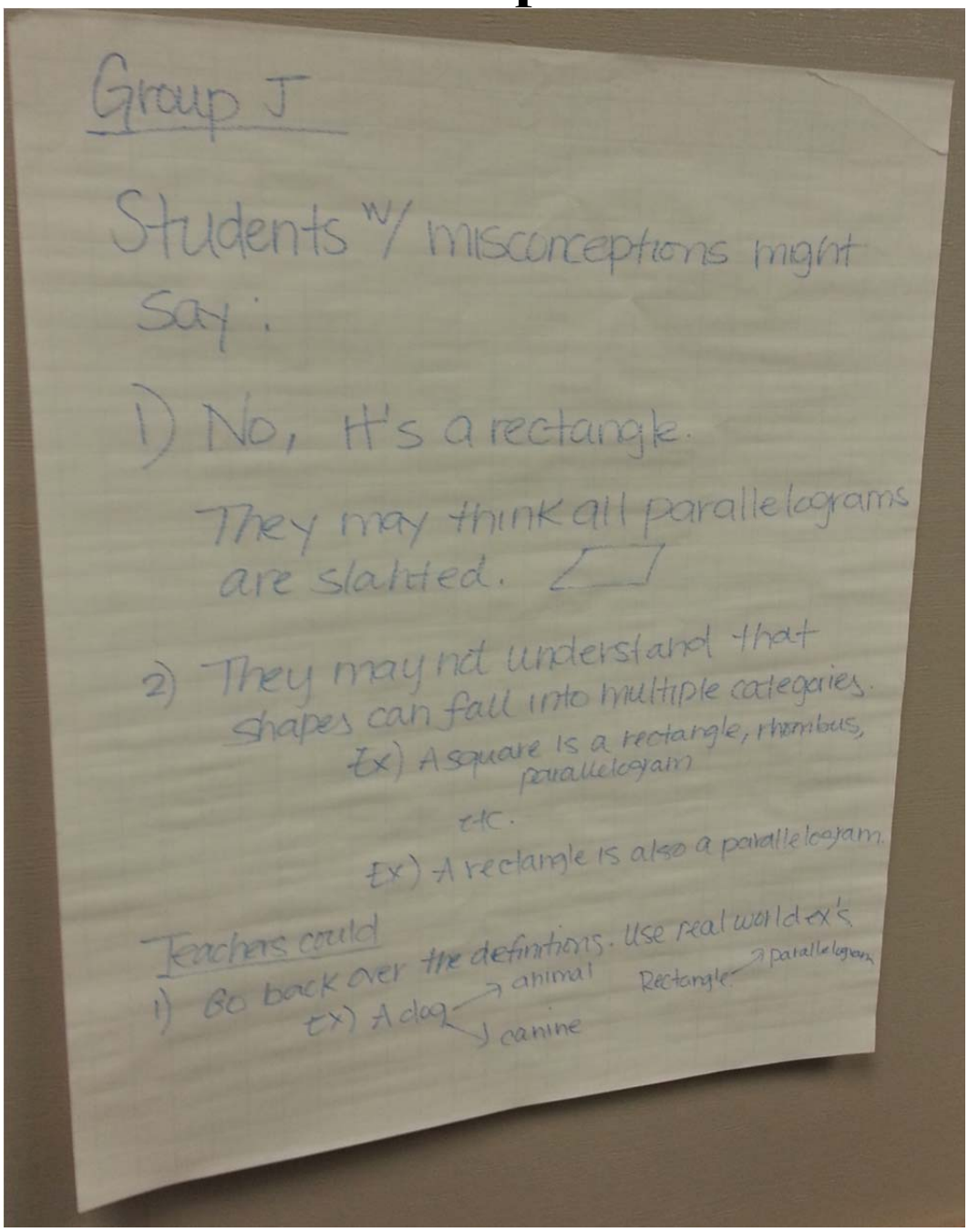
> incorrect proportion

$\frac{1}{3} = \frac{3}{x}$ $\frac{1}{3} = \frac{3}{5}$?

2) Yes! Trial & Error

3) · Questioning Strategies
· Explain Steps
· Reflect on variable meaning

Group J



Group J

Students w/ misconceptions might say:

1) No, it's a rectangle.

They may think all parallelograms are slanted.

2) They may not understand that shapes can fall into multiple categories.

Ex) A square is a rectangle, rhombus, parallelogram

etc.

Ex) A rectangle is also a parallelogram.

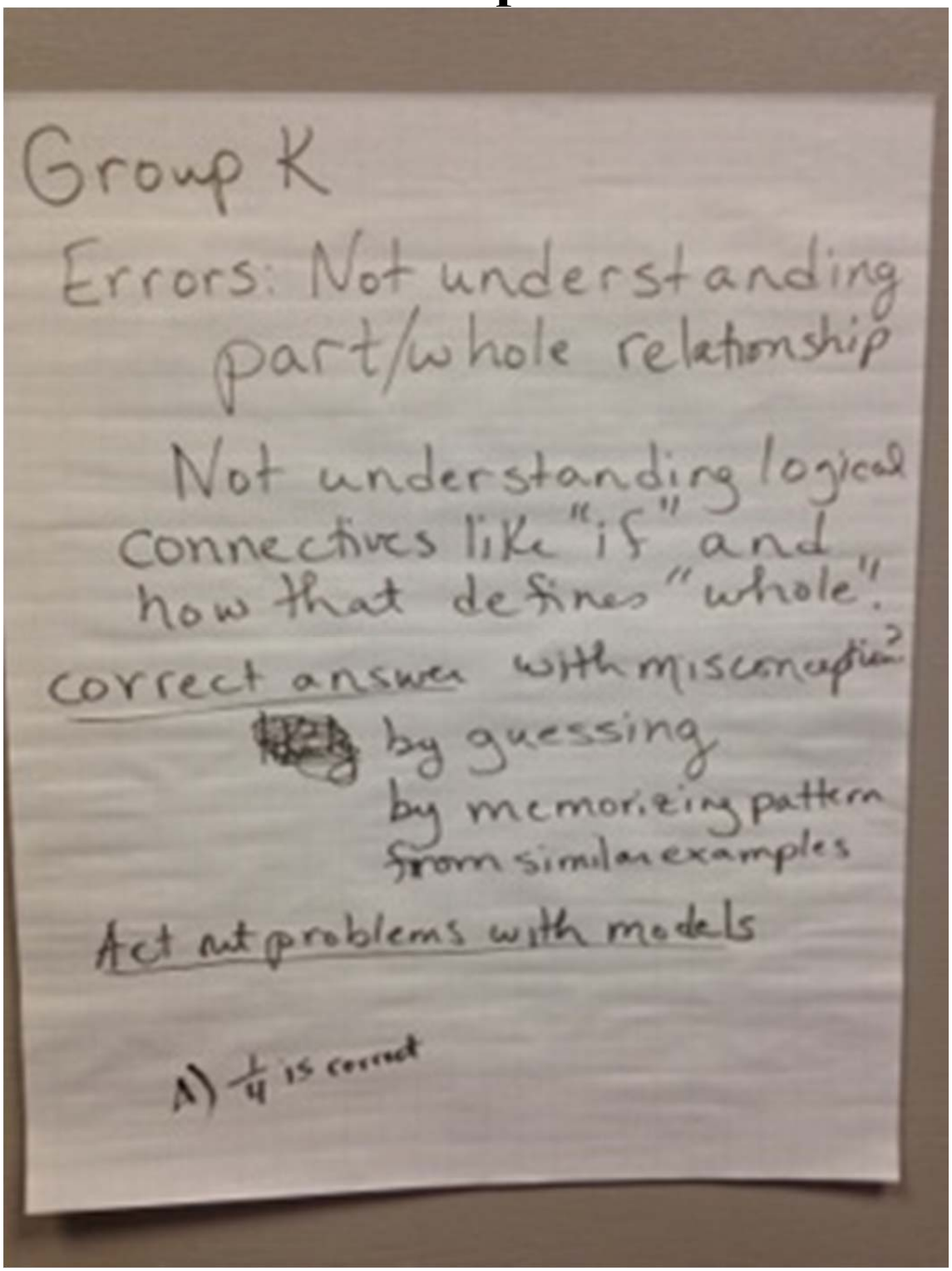
Teachers could

1) Go back over the definitions. Use real world ex's.

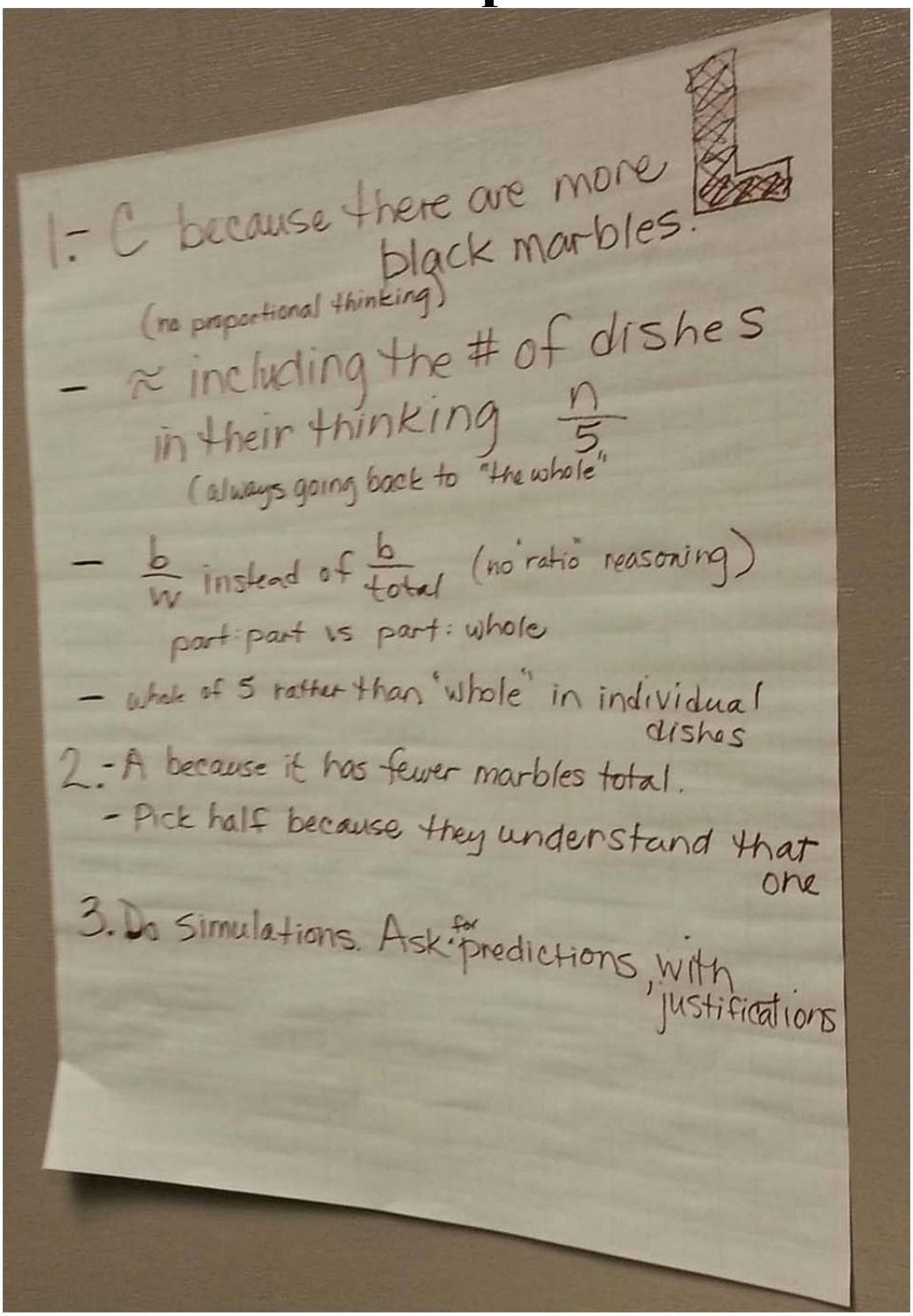
Ex) A dog → animal, → canine

Rectangle → parallelogram

Group K


Group K
Errors: Not understanding part/whole relationship
Not understanding logical connectives like "if" and how that defines "whole".
correct answer with misconception?
by guessing
by memorizing pattern from similar examples
Act out problems with models
A) 1/4 is correct

Group L



1.- C because there are more black marbles.

(no proportional thinking)

- ≈ including the # of dishes in their thinking $\frac{n}{5}$

(always going back to "the whole"

- $\frac{b}{w}$ instead of $\frac{b}{total}$ (no "ratio" reasoning)

part:part vs part:whole

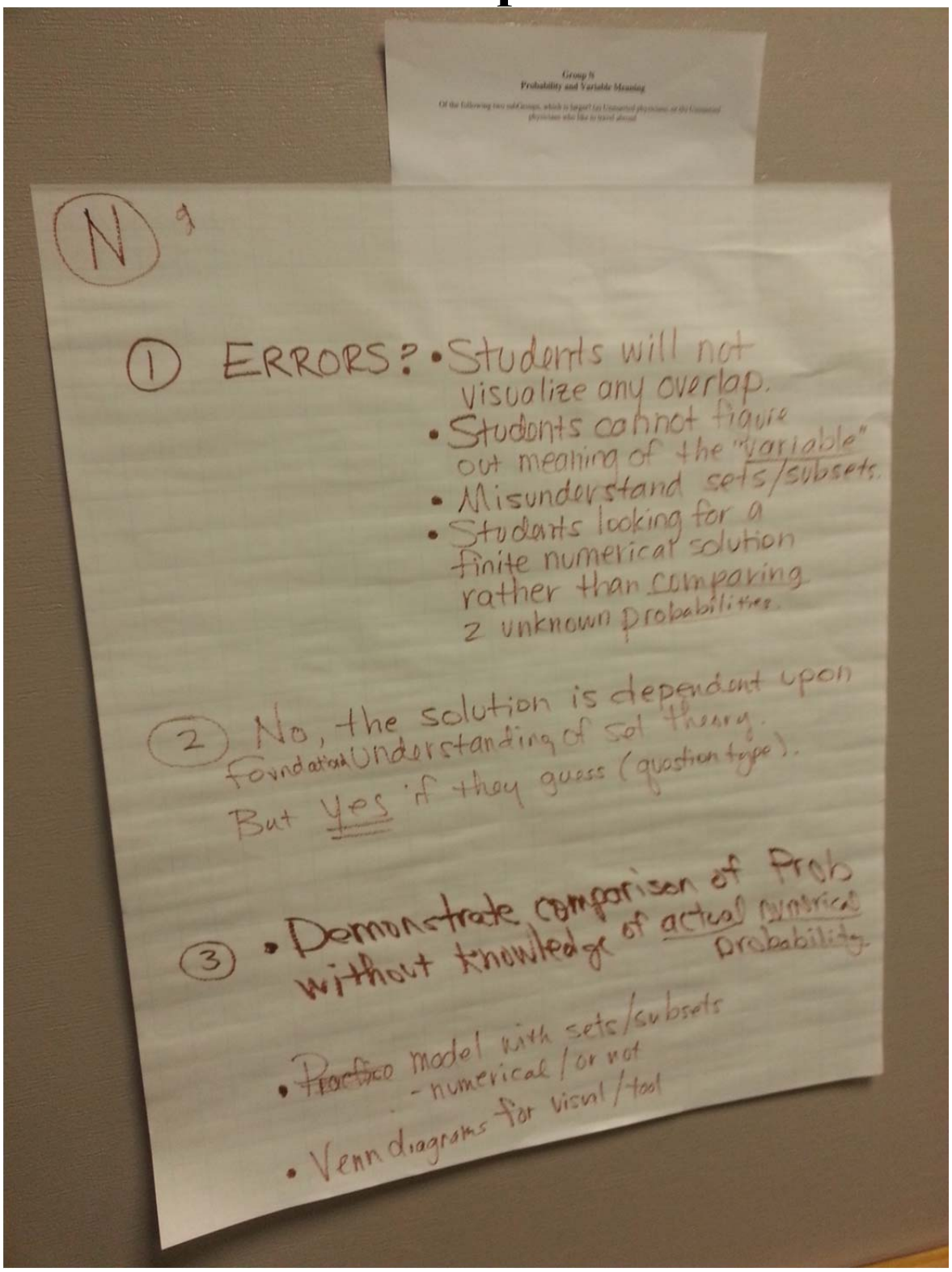
- whole of 5 rather than "whole" in individual dishes

2.- A because it has fewer marbles total.

- Pick half because they understand that one

3. Do Simulations. Ask for predictions, with justifications

Group N



Group N
Probability and Variable Meaning

Of the following two subGroups, which is larger? (a) Unmarried physicians, or (b) Unmarried physicians who like to travel abroad

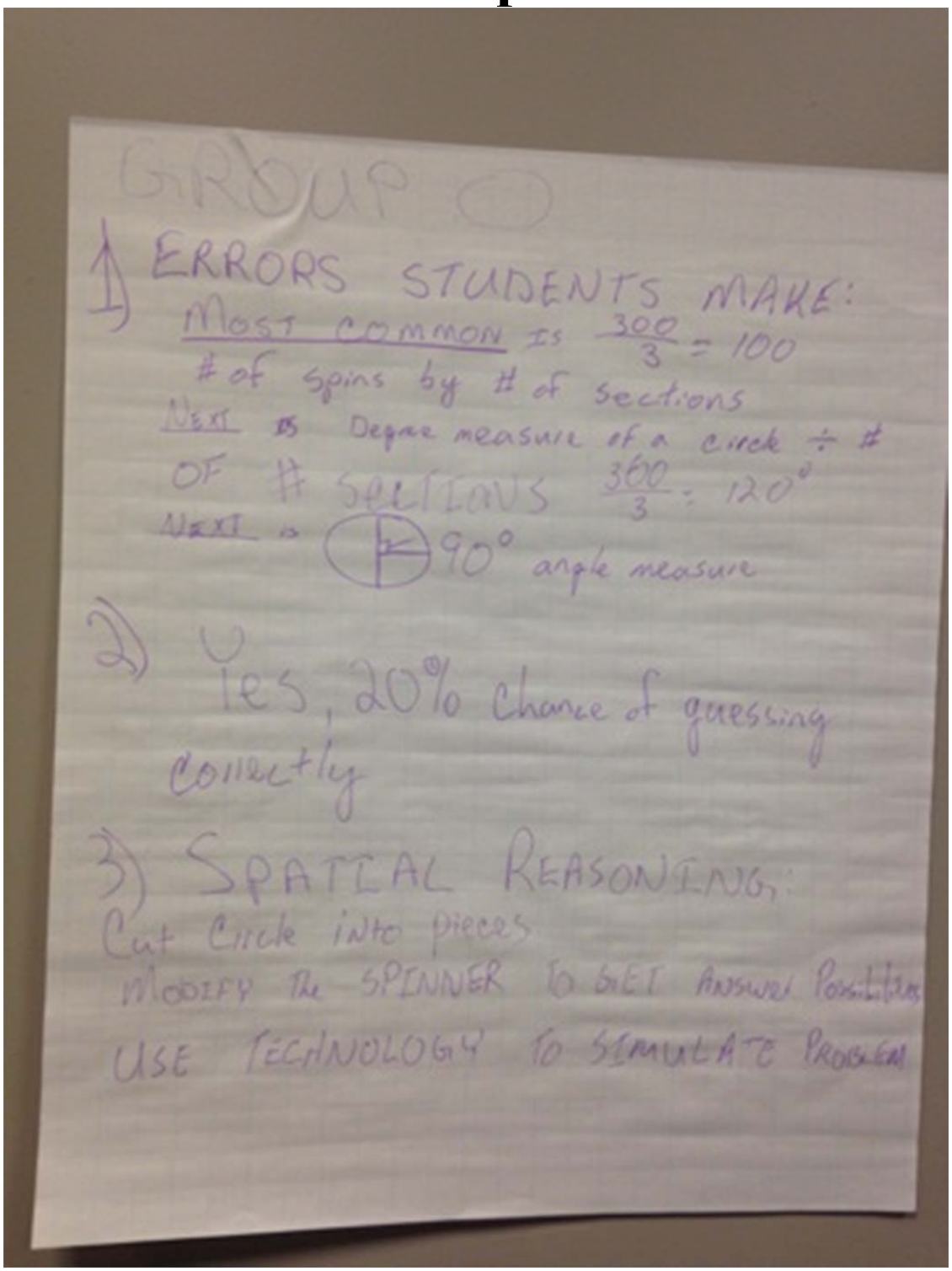
N 9

① ERRORS?
- Students will not visualize any overlap.
- Students cannot figure out meaning of the "variable"
- Misunderstand sets/subsets.
- Students looking for a finite numerical solution rather than comparing 2 unknown probabilities.

② No, the solution is dependent upon foundational understanding of set theory. But yes if they guess (question type).

③
- Demonstrate comparison of Prob without knowledge of actual numerical probability.
- ~~Practice~~ model with sets/subsets - numerical / or not
- Venn diagrams for visual / tool

Group O



GROUP O

1) ERRORS STUDENTS MAKE:

MOST COMMON IS $\frac{300}{3} = 100$

# of spins by # of sections

NEXT IS Degree measure of a circle ÷ #

OF # SECTIONS $\frac{360}{3} = 120°$

NEXT IS 90° angle measure

2) Yes, 20% Chance of guessing correctly

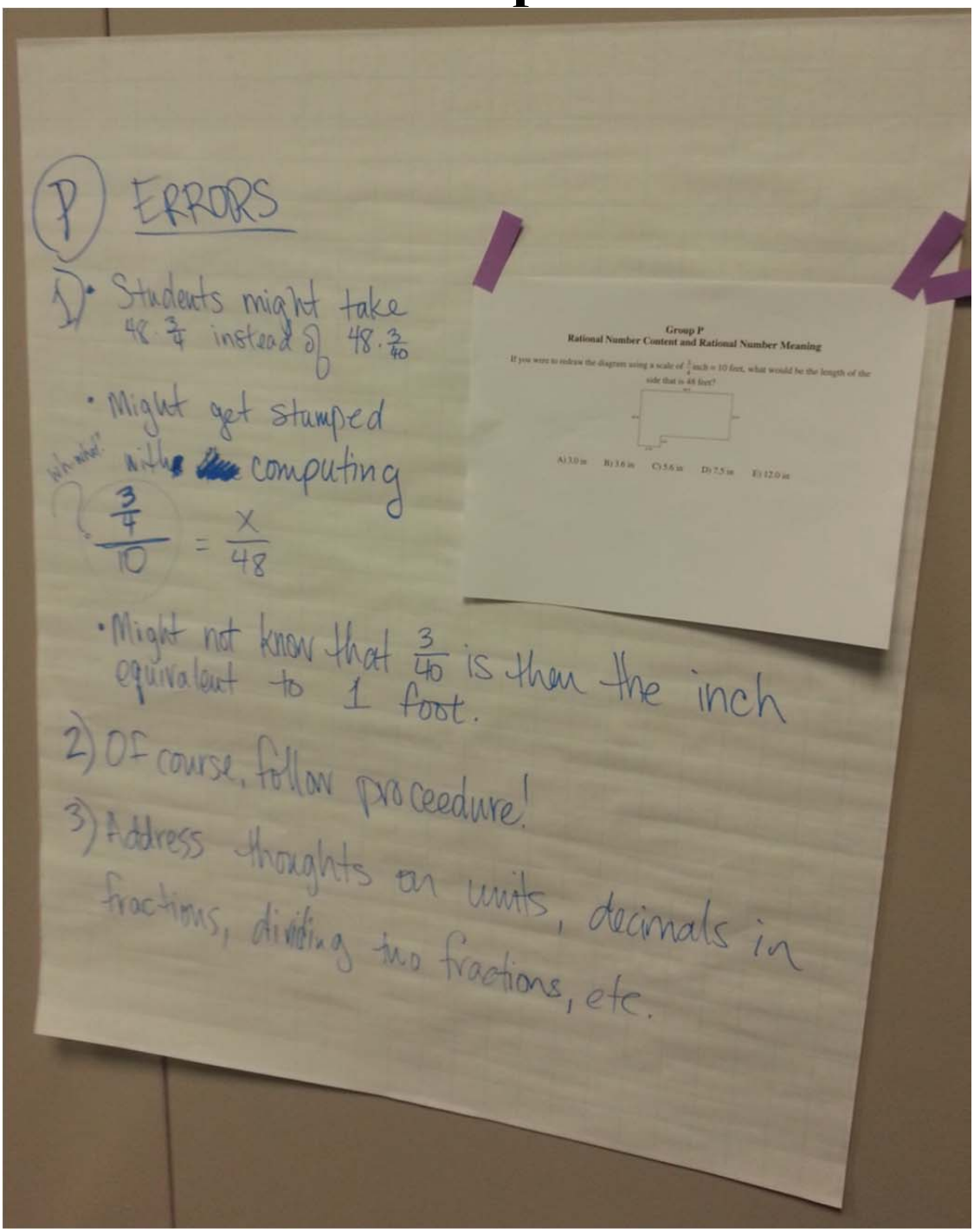
3) SPATIAL REASONING:

Cut Circle into pieces

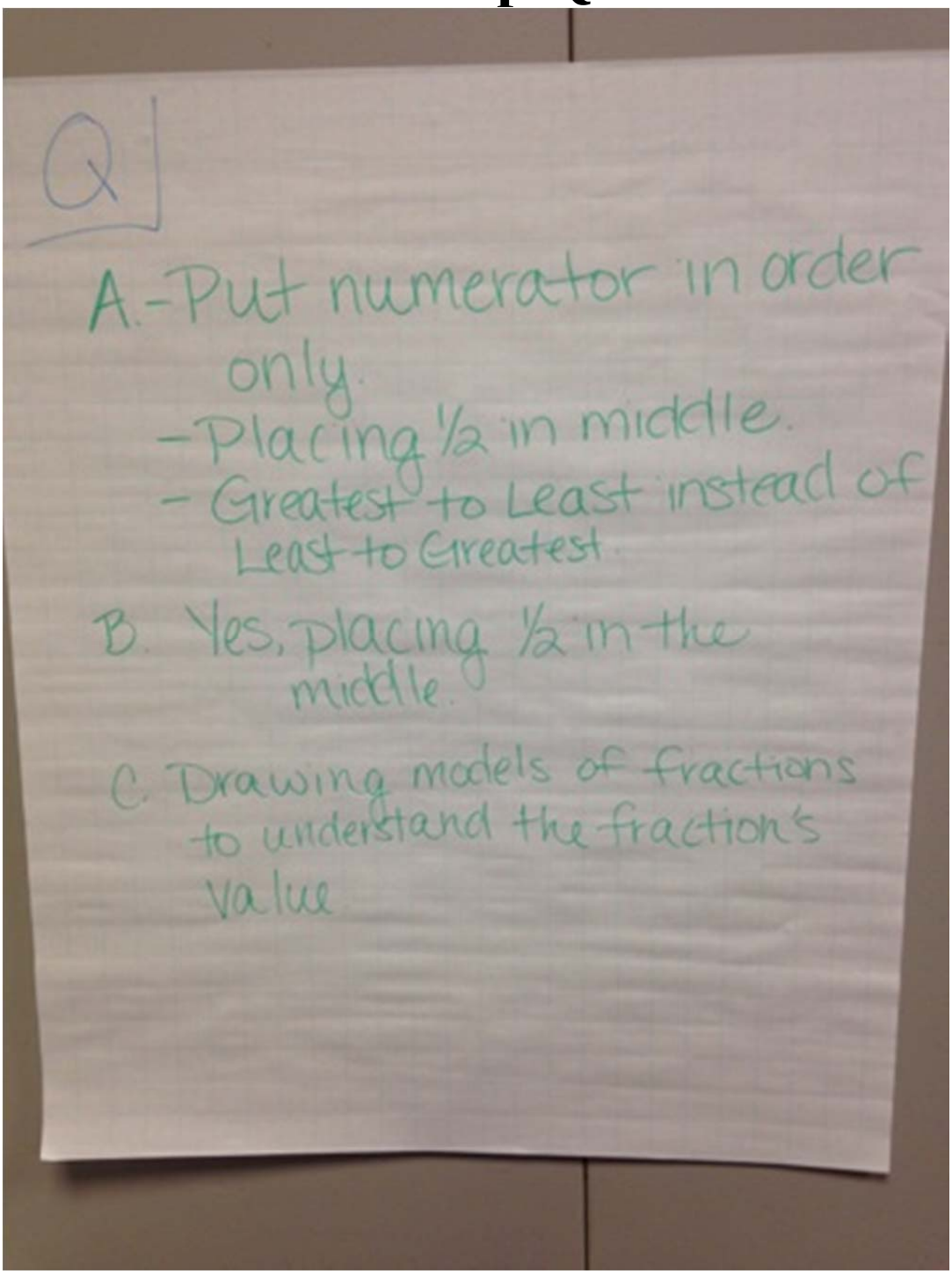
MODIFY The SPINNER TO GET ANSWER POSSIBILITIES

USE TECHNOLOGY TO SIMULATE PROBLEM

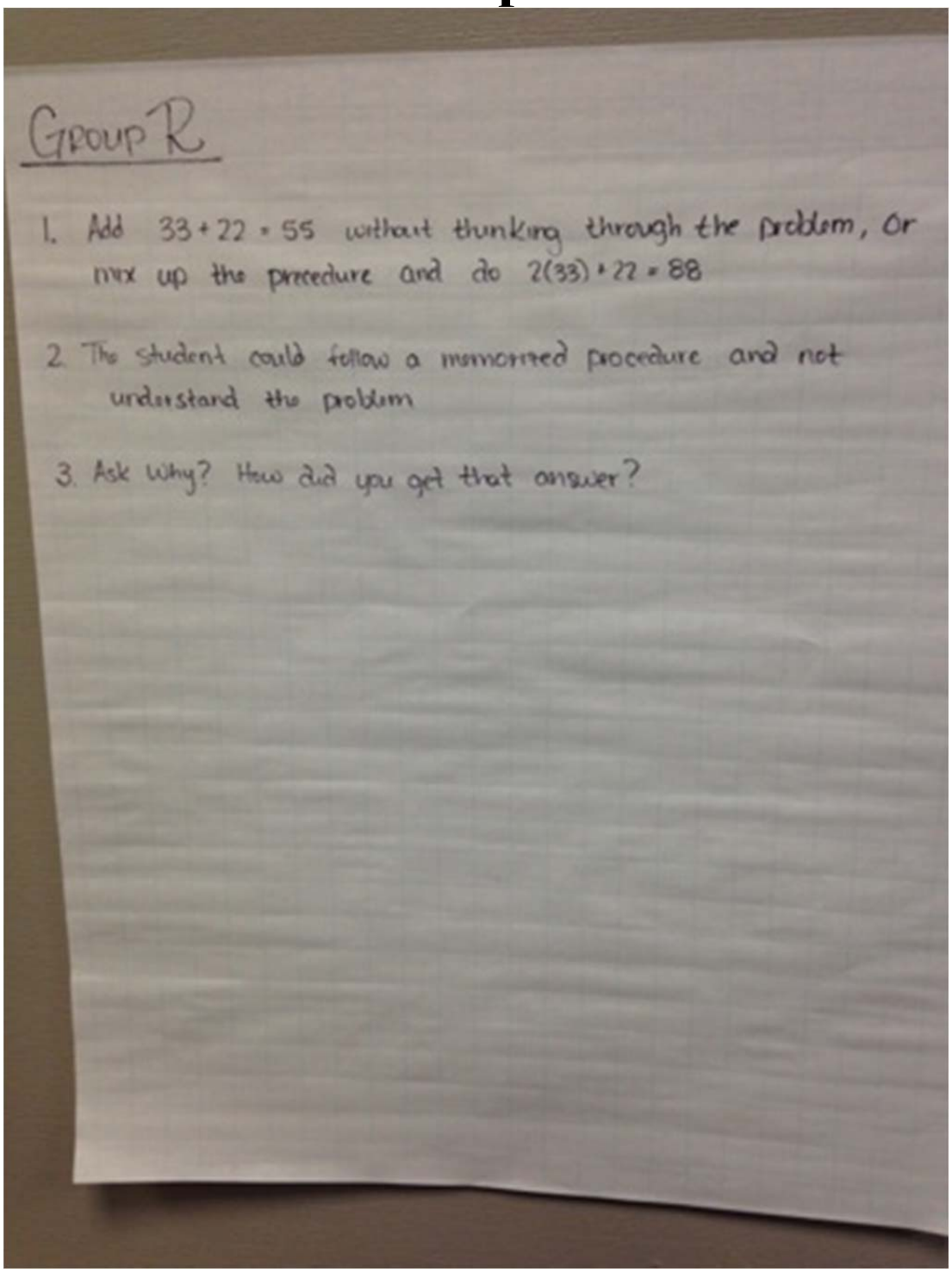
Group P


P ERRORS
1) • Students might take 48 · 3/4 instead of 48 · 3/40
• Might get stumped with computing
3/4 / 10 = x / 48
• Might not know that 3/40 is then the inch equivalent to 1 foot.
2) Of course, follow proceedure!
3) Address thoughts on units, decimals in fractions, dividing two fractions, etc.
Group P
Rational Number Content and Rational Number Meaning
If you were to redraw the diagram using a scale of 3/4 inch = 10 feet, what would be the length of the side that is 48 feet?
A) 3.0 in
B) 3.6 in
C) 5.6 in
D) 7.5 in
E) 12.0 in

Group Q


Q
A.-Put numerator in order only.
-Placing ½ in middle.
- Greatest to Least instead of Least to Greatest.
B. Yes, placing ½ in the middle.
C. Drawing models of fractions to understand the fraction's value

Group R



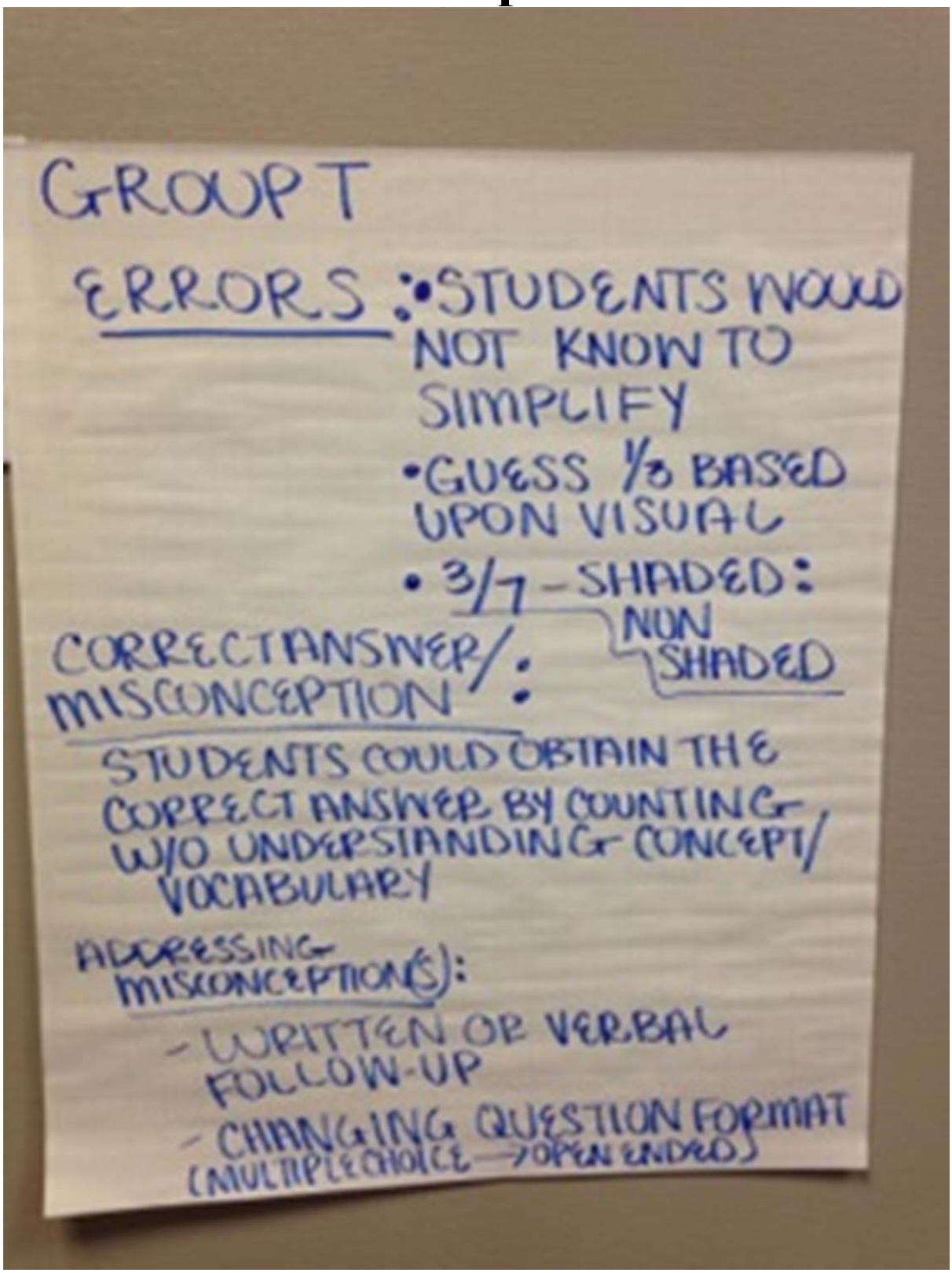
Group R

1. Add 33 + 22 = 55 without thinking through the problem, Or mix up the procedure and do 2(33) + 22 = 88

2. The student could follow a memorized procedure and not understand the problem

3. Ask Why? How did you get that answer?

Group T


GROUP T
ERRORS:
• STUDENTS WOULD NOT KNOW TO SIMPLIFY
• GUESS 1/3 BASED UPON VISUAL
• 3/7 - SHADED: NON SHADED
CORRECT ANSWER/ MISCONCEPTION:
STUDENTS COULD OBTAIN THE CORRECT ANSWER BY COUNTING W/O UNDERSTANDING CONCEPT/ VOCABULARY
ADDRESSING MISCONCEPTION(S):
- WRITTEN OR VERBAL FOLLOW-UP
- CHANGING QUESTION FORMAT (MULTIPLE CHOICE → OPEN ENDED)