Understanding and Addressing Algebra, Geometry, Rational Number, and Probability Misconceptions

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Session Objectives

Participants will:
• Learn about the underlying structure linking misconceptions across four content areas.
• Compare suggested pedagogical approaches with their current practice.
• Make connections between suggested pedagogical approaches for addressing misconceptions with CCSSM student learning objectives.
Differentiating the source of errors carries important consequences for intervention effectiveness (Konold, 1988)

Learning Mathematics with Understanding

- Conceptual & Procedural Knowledge (Hiebert & Carpenter, 1992)

  Concepts = Ideas, Facts

  Procedures = Skills, Algorithms

- Relational/Instrumental Understanding (Skemp, 1976/2006)
  - **Instrumental**: “Learning an increasing number of fixed plans, by which pupils can find their way from particular starting points (the data) to required finishing points (the answers to the questions)” (p. 14).
  - **Relational**: Building a conceptual structure (“schema”) that can be used to produce an unlimited number of plans for getting from an starting point within the schema to any finishing point.
Internal and External Understanding

• **Internal Understanding**: “A mathematical idea or procedure or fact is understood if it is part of an internal network... if its mental representation is part of a network of representations” (Hiebert & Carpenter, 1992, p. 67).

• **External Understanding**: Connections between different representations of the same mathematical idea, fact, or procedure or the same representation of different ideas, facts, or procedures. (e.g., spoken language, written symbols, algebra blocks, graphs, tables)

What does understanding look like in the classroom?

Understanding in the Classroom

(Clement, 1982; Fisher, 1988; Resnick, 1983; Rosnick & Clement, 1979)

<table>
<thead>
<tr>
<th>Observable Output to be Assessed</th>
<th>Underlying, Supporting Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct Solution</td>
<td>Correct Reasoning Applied Correctly to the Context</td>
</tr>
<tr>
<td>Correct Solution</td>
<td>Correct Reasoning Applied Incorrectly to the Context (or unrelated to context)</td>
</tr>
<tr>
<td>Correct Solution</td>
<td>Faulty Reasoning</td>
</tr>
<tr>
<td>Incorrect Solution</td>
<td>Correct Reasoning Applied Correctly to the Context (Careless Errors, Slips)</td>
</tr>
<tr>
<td>Incorrect Solution</td>
<td>Correct Reasoning Applied Incorrectly to the Context (i.e., Procedural Knowledge isolated from Conceptual Knowledge)</td>
</tr>
<tr>
<td>Incorrect Solution</td>
<td>Faulty Reasoning</td>
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An Example
(Lobato et al., 2010)

• *The clown walks 10 centimeters in 4 seconds. How far will the frog walk in 8 seconds if the frog travels at the same speed as the clown?*

• Students might be able to set up \( \frac{10}{4} = \frac{x}{8} \) and even solve correctly to find that \( x = 20 \) *without having a clear understanding of ratio.*

Deeper probing has found that some students who obtain a correct answer believe that the frog is traveling faster than the clown because it is going further. Such a misconception leaves students unprepared for problems stated in an unfamiliar format or more complex situations.

**CCSSM Standards of Practice**

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning.

“Rigor” in the Common Core

Foundational Structures

- Images that support thinking and reasoning about quantitative relationships (Thompson, 1994).
- Critical for developing mental operations (i.e., thinking)
- Represent the highest level of Piaget’s three types of images that people form as they develop understanding about a particular topic (Piaget & Inhelder, 1967).
Five Foundational Structures
(Rakes, 2010; Rakes & Ronau, under review)

- Rational Number Meaning (e.g., ratio, operator, quotient)
- Absolute/Relative Comparison (e.g., comparing ratios of quantities instead of quantities)
- Additive/Multiplicative Structures (e.g., recursive vs. functional relationships)
- Variable Meaning (e.g., beyond solving, what does the quantity represent?)
- Spatial Reasoning (e.g., van Hiele levels)

Foundational Structures in Learning Trajectories
(Lobato et al., 2010) (To view the Common Core Learning Trajectory Map, go to http://www.turnonccmath.net/

1. (1) Ratio as Composed Units
2. (2) Ratio as Multiplicative Comparisons
3. (3) Proportion as set of equivalent ratios (Absolute/Relative Comparisons)
4. (4) Students use iteration and partitioning to compose infinite sets of equivalent proportions
5. (5) Students use multiplication to abbreviate the process of iteration. $15 = 3 + 3 + 3 + 3 + 3 = 5$ groups of 3
6. (6) Students recognize that a set of composed units are related through multiplication.
7. (7) Rate
8. (8) Students use Reasoning and sense making to develop a set of algorithms for solving proportion problems
9. (9) Students connect linearity to proportionality, that linear equations of the form $y = mx$ are statements of proportionality.
10. (10) Students recognize that $\frac{\frac{1}{2}}{\frac{1}{3}}$ is the same as $\frac{1}{2} \times \frac{3}{1}$ths.
Foundational Structures in Learning Trajectories

(1) Ratio as Composed Units
Students recognize that every ratio is part of an infinite set of equivalent ratios, e.g., $1/3 = 2/6 = 3/9$, etc.

(2) Ratio as Multiplicative Comparisons
Students understand that multiplication is more than repeated addition. $3 \times 2 \rightarrow$ adding 2 three times $\rightarrow$ 3 twos.

(3) Proportion as set of equivalent ratios (Absolute/Relative Comparisons)
Students understand which quantities are equal in a proportion and interpret ratios as fractions ($2:3$ is same as $2/3$ of 1)

Example Pedagogical Strategies

• Targeting underlying reasoning
  – Questioning Strategies (Why? How Do You Know?)
  – Discussion Strategies (Think-Pair-Share)

• Assessment (Formative and Summative)
  – Tasks (all grade levels) target higher order thinking and high relevance/authenticity. (e.g., which is better rather than compute two values out of context)
## Review of Pedagogical Strategies

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<td>Formative and Summative Assessment</td>
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<td>Fostering Higher Order Thinking in Authentic Context</td>
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<tr>
<td>Think-Pair-Share</td>
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### Collaborative Groups

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<td>Probability</td>
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<td>M</td>
<td>N</td>
</tr>
<tr>
<td>Rational Number</td>
<td>P</td>
<td>Q</td>
<td>R</td>
<td>S</td>
</tr>
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- Introduce Yourself
- Read the Prompt
- Think about your responses individually to the following questions:
  1. What errors might students make on the problem? Can these errors be traced back to a misconception about the foundational structure? Why or why not?
  2. Is it possible for students to obtain a correct answer on the problem while having a misconception about the foundational structure? Why or why not?
  3. How might the teacher recognize and help students address the misconception(s)?
What? So What? Now What?
(Silver, Strong, Perini, 2001, p. 156)

• Teacher provides and experience for students:
  – Students describe WHAT happened during the experience
  – Students ask themselves SO WHAT? and reflect on the learning from the experience
  – Students ask themselves NOW WHAT? and consider ways to extend the learning to other situations
Explaining Solutions on Word Problems
(Silver, Strong, Perini, 2001, p. 156)

1. Teacher selects word problem
2. Students read the problem carefully, underlining the question and the information needed to develop a solution.
3. Students solve the problem.
4. Students reflect on the process and list, in chronological order, the steps they followed.
5. With the teacher’s help, students use transitional words (e.g., first, next, then, finally) to convert their steps into a paragraph. (Over time, students become independent.)
6. Students reread their explanations and ask:
   1. Are all the steps included?
   2. Are the steps in order?
   3. Do any terms need to be clarified?
   4. Are the transitional words well-chosen?
7. Students revise their explanations if necessary.

References


Reprinted in Mathematics Teaching in the Middle School, 12, 88-95.


Group A
Algebra and Rational Number Meaning

The temperature in degrees Celsius can be found by subtracting 32 from the temperature in degrees Fahrenheit and multiplying the result by \( \frac{5}{9} \). If the temperature of a furnace is 393 degrees Fahrenheit, what is it in degrees Celsius, to the nearest degree?

A) 650  
B) 1805  
C) 40  
D) 201  
E) 72

Group B
Algebra and Absolute/Relative Comparisons

A flea, usually less than an eighth of an inch long, can long jump about thirteen inches and high jump about eight inches. For a human 5.5 feet tall, what is the minimum distance needed to jump proportionally at least as far as a flea?

Group C
Algebra and Additive/Multiplicative Structures

In the equation \( y = 4x \), if the value of \( x \) is increased by 2, what is the effect on the value of \( y \)?

A) It is 8 more than the original amount.  
B) It is 6 more than the original amount.  
C) It is 2 more than the original amount.  
D) It is 16 times the original amount.  
E) It is 8 times the original amount.

Group D
Algebra and Variable Meaning

Cakes cost \( c \) dollars each and buns cost \( b \) dollars each. If I buy 4 cakes and 3 buns, what does \( 4c + 3b \) stand for?

Group E
Algebra and Spatial Reasoning

The figure below shows two right angles. The length of \( AE \) is \( x \) and the length of \( DE \) is 40. Show all of the steps that lead to finding the value of \( x \). Your last step should give the value of \( x \).

![Diagram](image)

Group F
Geometry and Absolute/Relative Comparisons

The figure below shows a shaded portion of a rectangle and another shaded portion of a circle. How might these two shaded portions be considered equal?
Group G

Geometry and Absolute/Relative Comparison

The figure below shows two right angles. The length of $AE$ is $x$ and the length of $DE$ is 40. Show all of the steps that lead to finding the value of $x$. Your last step should give the value of $x$.

![Figure with right angles and lengths labeled $x$, 3, 40, C, D]

Group H

Geometry and Additive/Multiplicative Structure

What is the area of the shaded figure?

A) 9 square centimeters
B) 11 square centimeters
C) 13 square centimeters
D) 14 square centimeters

Group I

Geometry and Variable Meaning

A scale drawing of a rectangular room is 5 inches by 3 inches. If 1 inch on this scale drawing represents 3 feet, what are the dimensions of the room?

A) 5 feet by 3 feet  
B) 5 feet by 9 feet  
C) 15 feet by 3 feet  
D) 15 feet by 5 feet  
E) 15 feet by 9 feet

Group J

Geometry and Spatial Reasoning

Sara was asked to draw a parallelogram. She drew the figure below.

Is Sara's figure a parallelogram? Why or why not?

Group K

Probability and Rational Number Meaning

The table below shows the gender and color of 7 puppies. If a puppy selected at random from the group is brown, what is the probability it is a male?

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>Black</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Brown</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
Group L
Probability and Absolute/Relative Comparison

A person is going to pick one marble without looking. For which dish is there the greatest probability of picking a black marble?

A) 4/12  B) 7/12  C) 3/12  D) 2/12  E) 3/12

Group M
Probability and Additive/Multiplicative Structure

Twelve skiers compete in the final round of the Olympic freestyle skiing competition. How many different top three winners are possible? (Gold, Silver, Bronze).

Group N
Probability and Variable Meaning

Of the following two subGroups, which is larger? (a) Unmarried physicians, or (b) Unmarried physicians who like to travel abroad.

Group O
Probability and Spatial Reasoning

If Rose spins a spinner like the one below 300 times, about how many times should she expect it to land on the space with a circle?

A) 75  B) 90  C) 100  D) 120  E) 150

Group P
Rational Number Content and Rational Number Meaning

If you were to redraw the diagram using a scale of \( \frac{3}{4} \) inch = 10 feet, what would be the length of the side that is 48 feet?

A) 3.0 in  B) 3.6 in  C) 5.6 in  D) 7.5 in  E) 12.0 in
Group Q
Rational Number and Absolute/Relative Comparison

In which of the following are the three fractions arranged from least to greatest?

A) \(\frac{2}{7}, \frac{1}{2}, \frac{5}{9}\)  
B) \(\frac{1}{2}, \frac{5}{9}, \frac{2}{7}\)  
C) \(\frac{1}{2}, \frac{5}{7}, \frac{2}{9}\)  
D) \(\frac{5}{2}, \frac{1}{7}, \frac{2}{9}\)  
E) \(\frac{5}{2}, \frac{1}{7}, \frac{2}{9}\)

Group R
Rational Number and Additive/Multiplicative Structures

The cost to mail a first-class letter is 33 cents for the first ounce. Each additional ounce costs 22 cents. (Fractions of an ounce are rounded up to the next whole ounce.)

How much would it cost to mail a letter that weighs 2.7 ounces?

A) 55 cents  
B) 66 cents  
C) 77 cents  
D) 88 cents  
E) 99 cents

Group S
Rational Number and Variable Meaning

Jo has two snakes, String Bean and Slim. Right now, String Bean is 4.1 feet long and Slim is 5.4 feet long. Jo knows that two years from now, both snakes will be fully grown...At her full length, String Bean will be 7.3 feet long, while Slim’s length when he is fully grown will be 8.6 feet. Over the next two years, will both snakes grow the same amount? Why or why not?

Group T
Rational Number and Spatial Reasoning

What fraction of the figure below is shaded?

A) \(\frac{1}{4}\)  
B) \(\frac{3}{10}\)  
C) \(\frac{1}{3}\)  
D) \(\frac{3}{7}\)  
E) \(\frac{7}{10}\)
Group A

A) Misconceptions

1) $393 - 32(\frac{5}{9})$ (Ans is $\frac{201}{2}$)

2) See phrase "subtract 32"
   See phrase "mutl by $\frac{5}{9}$"

3) does not see the "big picture"

2) May choose D because
   $393^\circ F$ is hot
   boiling $H_2O$ is $212^\circ$
   or $\frac{5}{9} \sim \frac{1}{2}$
   $\frac{1}{2}(393) \sim 200 \rightarrow$ choose D

3) Needs to write equation
   $C = \frac{5}{9}(F-32)$
   Interpret the word problem.
Group B

3) **Big Idea**
   - Comparisons
   - Yardstick
   - What is being compared?
   - Drawing → Visual
   - Kinesthetic
   - Spatial

   Relate to Movie:
   - Shrunken the Kids
   - Shrunken the Kids
Group C


Algebra & Additive/Multiplicative Structure

1] A is correct.
   B: increased so they add 2
   C: Not reading the total problem just adding 2 to coeff.
   D: 4 squared
   E: Multiplied by 2 instead of replacing 2

2] Memorize the steps but don't understand OR and don't understand how it relates to an increase

3] To address: graphs, charts... allow students to make connections between numerical pattern & algebraic representation leads to better algebraic reasoning
GROUP D

Algebra + Variable Meaning

Correct answer: $4c + 3b$ stands for the total cost of 4 cakes and 3 buns.

#1 $c = \#$ of cakes  $b = \#$ of buns

#2 not understanding variable meaning

Combine variable terms to get 7

4c is not recognized as $4 \cdot c$,
ignore variables.

#2 Yes could say it's total cost w/ out understanding the meaning of the variables as a rate.
not understanding rational # meaning.
Group E

1. Errors
   - Not viewing $AB$ as $40 + x$, but as $40$
     \[ \frac{9}{3} = \frac{40}{x} \]
     instead of \[ \frac{9}{3} = \frac{40 + x}{x} \]
   - Viewing $AB$ as $40x$ instead of $40 + x$
   - Using additive reasoning instead of multiplicative to solve the proportion

2. We didn’t see a way to get to the correct answer with these misconceptions.

3. [Diagram with triangles and variables]
Group F

1. Students will struggle w/the proportional relationship. The terms equal & proportional will be confusing.
   
   Ex: \( \frac{3}{8} \) of gallon vs. \( \frac{3}{8} \) of swimming pool (water in)
   
   The quantity may not be equal.

2) Students will say "The proportion is same so must be equal"

3) Concrete examples of simpler situations
   building frames, models, visuals
   manipulatives, paper plates
Group G

* triangles are similar

\[ \frac{3}{9} = \frac{x}{40} \]

Solving using algebraic proportions:

\[ 10.3 = 9x \]

Collect evidence - measured frequency

\[ \frac{3}{x} = \frac{40}{9} \rightarrow x = 100 \]

\[ 9.3 = 40 - x \]

Let's do bivariate relative comparison

\[ 3x = 9 \times 10 \rightarrow x = 30 \]

\[ \frac{3}{0.9} = \frac{x}{40} \rightarrow x = 13.33 \]
Group H:

0. Errors:
   - Perimeter vs. area
   - Counting only "whole" squares
   - Counting all squares as "whole"

1. Yes — if you only count the bold perimeter, you can still get 11

2. The shaded fig. + multiple choice make it easy to guess correctly without understanding the concept of area.

3. Modeling area vs. perimeter
   - Discuss grouping of "wholes" vs. "half"
   - Tangible examples?
1) Don’t realize both dimensions need to be multiplied by 3.

\[ K = \frac{1}{3} \text{ or } \frac{3}{1} \text{ which way?} \]

2) Incorrect proportion

\[ \frac{1}{3} = \frac{3}{x} \quad \frac{1}{3} = \frac{3}{5} \]

3) Yes! Trial & Error

3) Questioning Strategies
   - Explain Steps
   - Reflect on Variable Meaning
Group J

Students’ misconceptions might say:

1) No, it’s a rectangle.
   They may think all parallelograms are slatted. [ ]

2) They may not understand that shapes can fall into multiple categories.
   Ex) A square is a rectangle, rhombus, parallelogram, etc.
   Ex) A rectangle is also a parallelogram.

Teachers could:
1) Go back over the definitions. Use real world ex’s.
   [Ex] A dog = animal, rectangle = parallelogram.

Jeanine
Group K

Errors: Not understanding part/whole relationship

Not understanding logical connectives like "if" and how that defines "whole". Correct answer with misconception by guessing by memorizing pattern from similar examples. Act out problems with models.

A) $\frac{1}{4}$ is correct
1. C because there are more black marbles.
   (no proportional thinking)
   - Including the # of dishes in their thinking \( \frac{n}{5} \)
   - Always going back to "the whole"

2. \( \frac{b}{w} \) instead of \( \frac{b}{\text{Total}} \) (no ratio reasoning)
   - Part:part vs part:whole
   - Whole of 5 rather than "whole" in individual dishes

2. A because it has fewer marbles total.
   - Pick half because they understand that one

3. Do simulations. Ask for predictions with justifications
Group N

1. **ERRORS?**
   - Students will not visualize any overlap.
   - Students cannot figure out meaning of the "variable".
   - Misunderstand sets/subsets.
   - Students looking for a finite numerical solution rather than comparing 2 unknown probabilities.

2. No, the solution is dependent upon fundamental understanding of set theory.
   But **yes** if they guess (question type).

3. Demonstrate comparison of prob without knowledge of actual numerical probability.
   - Practice model with sets/subsets - numerical/or not
   - Venn diagrams for visual/aid
Group O

1. ERRORS STUDENTS MAKE:
   Most common is $\frac{300}{3} = 100$
   # of spins by # of sections
   Next is degree measure of a circle ÷ #
   OF # SECTIONS $\frac{360}{3} = 120^\circ$
   Next is $\odot 90^\circ$ angle measure

2. Yes, 20% chance of guessing correctly

3. SPATIAL REASONING:
   Cut circle into pieces
   Modify the spinner to get answer possible
   Use technology to simulate problem
1. **Errors**

   Students might take 48 3/10 instead of 48 3/10.

   - Might get confused with computing: \( \frac{3}{10} = \frac{x}{48} \)

   - Might not know that \( \frac{3}{10} \) is the inch equivalent to 1 foot.

2. Of course, follow procedure!

3. Address thoughts on units, decimals in fractions, dividing two fractions, etc.
Group Q

Q

A. Put numerator in order only.
   - Placing 1/2 in middle.
   - Greatest to Least instead of Least to Greatest.

B. Yes, placing 1/2 in the middle.

C. Drawing models of fractions to understand the fraction's value.
Group R

1. Add $33 + 22 = 55$ without thinking through the problem, or
   mix up the procedure and do $2(33) + 22 = 88$

2. The student could follow a memorized procedure and not understand the problem

3. Ask Why? How did you get that answer?
GROUP T

ERRORS:
- Students would not know to simplify.
- Guess ⅔ based upon visual.
- 3/7 - SHADEd:
  CORRECT ANSWER: NON SHADEd
MISCONCEPTION: STUDENTS COULD OBTAIN THE CORRECT ANSWER BY COUNTING W/O UNDERSTANDING CONCEPT/VOCABULARY

ADDRESSING MISCONCEPTION(S):
- Written or verbal follow-up.
- Changing question format (multiple choice → open ended)